Soft Pion Emission in Hard Exclusive Pion Production

Maxim V. Polyakov[a,b], Simone Stratmann[b]

6th September 2006

[a] *St. Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg, Russia*
[b] *Institut für Theoretische Physik II, Ruhr-Universität, D-44780 Bochum, Germany*

Abstract

We investigate hard exclusive reactions on the nucleon with soft pion emission. A parametrization of corresponding hadronic matrix elements in terms of parton distributions for final pion-nucleon state is provided. These distributions are calculated in terms of nucleon and pion GPDs and the pion distribution amplitude via soft-pion theorems. Some observables for the process of hard charged pion production on the proton with soft pion emission are computed.

1 Motivation and outline

The field of hard exclusive reactions has been studied intensively during the past decade. The investigations have proceeded in the theoretical as well as in the experimental sector; reviews are given e.g. in references [1, 2, 3, 4, 5]. Two prominent representatives of the hard exclusive processes are deeply virtual Compton scattering (DVCS) and hard meson production (HMP). The nucleon properties which enter these reactions are formulated in terms of generalized parton distributions (GPDs). On the one hand, these functions can be viewed as generalizations of the usual forward parton distributions, on the other hand they are directly related to nucleon form factors through their moments. So it has been argued that hard exclusive reactions can provide useful new insights into the partonic nucleon structure which are not accessible through the usual electroweak probes. In this context, for example, the form factors of the energy-momentum tensor have been discussed, see e.g. references [6, 7]. Additionally, in the case of meson production, one can obtain information about the involved distributions amplitudes.

In this article, we investigate the situation when in a hard exclusive reaction instead of the final nucleon a nucleon-pion state with low invariant mass appears. Since it is produced close to the threshold, the pion is denoted as *soft*. On the experimental side, a separation of a particular hard reaction with and without soft pion emission cannot always be guaranteed. In this sense, the process with soft pion can be viewed as a contamination of the fully exclusive DVCS or HMP, and theoretical estimates about this disturbing background are desirable.

Apart from such practical considerations, these new reactions are worth being studied in their own right. They provide an opportunity to investigate soft pion emission from the nucleon induced by nonlocal lightcone operators as opposed to the local vector or axial operators to which we are restricted in usual electroweak pion production. Therefore, analogously to how a soft process such as pion-electroproduction can provide information about nucleon form factors, hard processes with soft pion emission might contribute to a better understanding of quantities such as generalized parton distributions.

Guichon et. al. have addressed this question for the process of DVCS with soft pion production [8]. For the calculation of the pertinent hadronic matrix element, they presented a soft-pion theorem based on current algebra and chiral symmetry. Moreover, they modeled the effect of the $\Delta(1232)$ resonance which is located not far from the pion production threshold. By this method, they could give predictions for certain cross sections and asymmetries. However, in all their considerations, the region of small momentum transfer was explicitely excluded.

In the following, we present our approach of calculating soft pion emission in hard exclusive reactions. Based on polology arguments, "PCAC", and certain properties of the chiral symmetry transformation, we derive corresponding soft-pion theorems. These differ from the results in [8] through additional pion pole terms. The implementation of these new contributions allows in particular to extend the region of applicability down to small momentum transfer. Using this improved result, we calculate the effect of soft pions in hard π^+ production off the proton.

The outline is as follows. Basic kinematical considerations are given in section 2. In section 3, we provide a parametrization of the matrix elements for pion emission induced by twist-2 lightcone operators. We denote the invariant functions that come up in this procedure as pion-nucleon (πN) parton distributions. These functions are the generalizations of GPDs for the case of pion emission. We discuss some of their properties, in particular their behavior at the pion threshold and the meaning of some moments. In section 4, we give a detailed derivation and the results of soft-pion theorems for several twist-2 operators. We check that in certain limiting cases our expressions are consistent with previous calculations. Finally, in section 5, we apply these results to the process of hard pion production off the proton with soft pion emission near threshold. The amplitude of the process for arbitrary isospins is given. Further, we provide the following numerical estimates for hard π^+ production: the transverse spin asymmetry of the process $\gamma^* + \mathrm{p} \to \pi^+ + N + \pi_{\mathrm{soft}}$ is calculated, and the contamination of the longitudinal cross section and the transverse spin asymmetry of the pure process $\gamma^* + \mathrm{p} \to \pi^+ + \mathrm{n}$ through soft-pion admixture is determined.

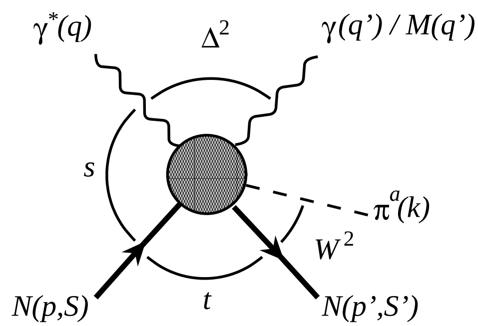

Figure 1: Particle momenta and invariants for the exclusive reaction with soft pion emission.

2 Kinematics

We consider the collision of a virtual photon γ^* with momentum q and a nucleon N with momentum p and spin S. In the final state, we have a nucleon with momentum p' and spin S', a pion π with momentum k and isospin a, and either a real photon γ (DVCS) or a specified meson M with momentum q':

$$\gamma^*(q) + N(p,S) \to N(p',S') + \pi^a(k) + \left\{ \begin{array}{c} \gamma(q') \\ M(q') \end{array} \right. .$$

It is useful to define an average momentum $\bar{p}$ and a momentum transfer Δ in the following way:

$$\bar{p} \equiv \frac{p+p'+k}{2}, \qquad \Delta \equiv q - q' = p' + k - p. \tag{1}$$

Further, we introduce the Lorentz invariants

$$s \equiv (p+q)^2, \qquad t \equiv (p-p')^2, \qquad u \equiv (p-k)^2, \qquad W^2 \equiv (p'+k)^2, \tag{2}$$

see also the illustration in figure 1 (we point to the fact that here Δ^2 is *not* identical to the nucleon momentum transfer t unlike in the case without pion). The kinematical region of hard scattering is characterized by a large photon virtuality Q^2, a large energy ν of the virtual photon in the target rest frame, and a fixed Bjorken variable x_B:

$$Q^2 \equiv -q^2,\ M\nu \equiv p\cdot q \quad \gg \quad -\Delta^2,\ -t,\ W^2,\ M^2,\ m_\pi^2,\ q'^2, \tag{3}$$

$$x_B \equiv \frac{Q^2}{2M\nu}. \tag{4}$$

Here, M and m_π denote the nucleon and pion mass, respectively.

We shall refer to the pion $\pi(k)$ as *soft* if it appears sufficiently close to its production threshold in the following sense. Directly at the threshold, the variables W^2 and u are fixed as

$$W_{\rm th}^2 = (M+m_\pi)^2, \qquad u_{\rm th} = (M-m_\pi)^2 + \frac{m_\pi}{M}t. \tag{5}$$

We allow for deviations from these values that are of the size

$$W^2 - W_{\rm th}^2 \lesssim M m_\pi, \qquad -(u-u_{\rm th}) \lesssim M m_\pi. \tag{6}$$

Moreover, we require for the nucleon momentum transfer that

$$-t \lesssim M^2. \tag{7}$$

Relations (5) to (7) settle the soft pion kinematics (note that they imply $-\Delta^2 \lesssim M^2$). Roughly speaking, we can summarize these conditions simply as

$$k = \mathcal{O}(m_\pi), \qquad p, p' = \mathcal{O}(M). \tag{8}$$

Let us now turn to a Sudakov decomposition of the particle momenta. Two lightcone vectors $\tilde{n}$ and n are defined in a frame where $\bar{p} = (\bar{p}_0, 0, 0, \bar{p}_z)$ and $q = (q_0, 0, 0, q_z)$ via

$$\tilde{n} = (\bar{p}_0 + \bar{p}_z)(1,0,0,1), \qquad n = \frac{1}{2(\bar{p}_0+\bar{p}_z)}(1,0,0,-1). \tag{9}$$

Consequently, we obtain the average momentum and the virtual photon momentum in the form

$$\bar{p} = \tilde{n} + \frac{\bar{M}^2}{2}n, \qquad q = -2\xi\tilde{n} + \frac{Q^2}{4\xi}n, \tag{10}$$

with

$$\bar{M}^2 = \bar{p}^2 = \frac{M^2+W^2}{2} - \frac{\Delta^2}{4}, \qquad -2\xi = \frac{\bar{p}\cdot q}{\bar{M}^2} - \sqrt{\left(\frac{\bar{p}\cdot q}{\bar{M}^2}\right)^2 + \frac{Q^2}{\bar{M}^2}}. \tag{11}$$

The decompositions of some other vectors to leading order in $1/Q^2$ read

$$\begin{aligned}
\Delta &= -2\xi\tilde{n} + \left(\bar{M}^2\xi + \frac{W^2-M^2}{2}\right)n + \Delta_\perp, &(12)\\
p &= (1+\xi)\tilde{n} + \left[\frac{\bar{M}^2}{2}(1-\xi) - \frac{W^2-M^2}{4}\right]n - \frac{\Delta_\perp}{2}, &(13)\\
p'+k &= (1-\xi)\tilde{n} + \left[\frac{\bar{M}^2}{2}(1+\xi) + \frac{W^2-M^2}{4}\right]n + \frac{\Delta_\perp}{2}. &(14)
\end{aligned}$$

Note that these expressions lead immediately to the relations

$$x_B = \frac{2\xi}{1+\xi} \tag{15}$$

and

$$\Delta_\perp^2 = -(1-\xi^2)(|\Delta^2| - |\Delta^2|_{\min}), \tag{16}$$

with the minimal value of the momentum transfer given by

$$|\Delta^2|_{\min} = \frac{2\xi}{1-\xi^2}[(W^2+M^2)\xi + W^2 - M^2] = \frac{x_B^2 M^2 + x_B(W^2-M^2)}{1-x_B}. \tag{17}$$

Setting $k=0$ and $W=M$ everywhere, one recovers the well-known formulas of the familiar case without pion.

Finally, in order to quantify how the final nucleon and pion share the momentum with respect to the lightcone direction $\tilde{n}$, we further introduce a variable α such that

$$k \cdot n = \alpha(1-\xi), \qquad p' \cdot n = (1-\alpha)(1-\xi). \tag{18}$$

3 Formulation of pion-nucleon parton distributions

3.1 General parametrization

In the case of ordinary DVCS or hard meson production, the factorization of the amplitude leads to nucleon matrix elements of lightcone operators. These nonperturbative objects are parametrized in terms of generalized parton distributions. For hard meson production, additionally the distribution amplitude of the final meson appears.

In the present situation, where we take into account an additional pion, we arrive at matrix elements of lightcone operators with initial nucleon and final pion-nucleon state. In dealing with these objects, our first step is to provide a parametrization. Let us start with the isoscalar quark operator of vector type:

$$\int \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{ß\lambda x \bar{p}\cdot n} \langle N(p')\pi^a(k)|\bar{\psi}(-\lambda n/2) \not{n} \psi(\lambda n/2)|N(p)\rangle = \frac{ßg_A}{M f_\pi} \sum_{i=1}^{4} \bar{U}(p')\, \Gamma_i H_i^{(0)} \tau^a U(p) \tag{19}$$

where we have introduced the Dirac matrices

$$\Gamma_1 = \gamma_5, \qquad \Gamma_2 = \frac{M\not{n}}{n\cdot\bar{p}}\gamma_5, \qquad \Gamma_3 = \frac{\not{k}}{M}\gamma_5, \qquad \Gamma_4 = \frac{\not{k}\not{n}}{n\cdot\bar{p}}\gamma_5 \tag{20}$$

with the definitions $\Delta = p' + k - p$ and $\bar{p} = (p'+k+p)/2$. The constant g_A is the isovector axial coupling constant, and f_π is the pion decay constant normalized according to the experimental value of 93 MeV. Including these constants into the definition will be convenient in later calculations. Here and in the following, the insertion of the appropriate gauge links within the operators is always understood. The functions $H_i^{(0)}$, that we shall call pion-nucleon parton distributions, depend on the momentum fraction x as well as on five further quantities that can be built from the vectors n, p, p', and k which we have at our disposal. We choose

$$H_i^{(0)} = H_i^{(0)}(x, \xi, \Delta^2, \alpha, t, W^2), \tag{21}$$

where in the present context of parametrization the definitions of the variables are

$$\xi = -\frac{n\cdot\Delta}{2n\cdot\bar{p}}, \qquad \alpha = \frac{n\cdot k}{n\cdot(p'+k)}, \qquad t=(p'-p)^2, \qquad W^2=(p'+k)^2. \tag{22}$$

In a similar way, we introduce the isovector even and odd distributions $H_i^{(+)}$ and $H_i^{(-)}$,

$$\begin{aligned}\int \frac{\mathrm{d}\lambda}{2\pi}\mathrm{e}^{\text{ß}\lambda x\bar{p}\cdot n}\langle N'\pi^a|\bar{\psi}(-\lambda n/2)\not{n}\tau^b\psi(\lambda n/2)|N\rangle \\ = \frac{\text{ß}g_A}{Mf_\pi}\sum_{i=1}^{4}\bar{U}'\Gamma_i(\delta^{ab}H_i^{(+)}+\text{ß}\varepsilon^{abc}\tau^c H_i^{(-)})U,\end{aligned} \tag{23}$$

which depend on the same set of arguments as $H^{(0)}$, of course.

The matrix elements of the quark operators of axial type differ from the previous ones only through the insertion of a matrix γ_5. Therefore, we can define in an analogous manner isoscalar distributions,

$$\int \frac{\mathrm{d}\lambda}{2\pi}\mathrm{e}^{\text{ß}\lambda x\bar{p}\cdot n}\langle N'\pi^a|\bar{\psi}(-\lambda n/2)\not{n}\gamma_5\psi(\lambda n/2)|N\rangle = \frac{\text{ß}g_A}{Mf_\pi}\sum_{i=1}^{4}\bar{U}'\Gamma_i\gamma_5\tau^a\tilde{H}_i^{(0)}U \tag{24}$$

and isovector distributions,

$$\begin{aligned}\int \frac{\mathrm{d}\lambda}{2\pi}\mathrm{e}^{\text{ß}\lambda x\bar{p}\cdot n}\langle N'\pi^a|\bar{\psi}(-\lambda n/2)\not{n}\gamma_5\tau^b\psi(\lambda n/2)|N\rangle \\ = \frac{\text{ß}g_A}{Mf_\pi}\sum_{i=1}^{4}\bar{U}'\Gamma_i\gamma_5(\delta^{ab}\tilde{H}_i^{(+)}+\text{ß}\varepsilon^{abc}\tau^c\tilde{H}_i^{(-)})U.\end{aligned} \tag{25}$$

Furthermore, the gluon matrix element can be parametrized as

$$\int \frac{\mathrm{d}\lambda}{2\pi}\mathrm{e}^{\text{ß}\lambda x\bar{p}\cdot n}\langle N'\pi^a|F^{\mu\rho}(-\lambda n/2)F_\rho{}^\nu(\lambda n/2)n_\mu n_\nu|N\rangle = \frac{\text{ß}g_A}{Mf_\pi}\frac{n\cdot\bar{p}}{2}\sum_{i=1}^{4}\bar{U}'\Gamma_i\tau^a xH_i^{(G)}U. \tag{26}$$

At this point we have to comment on a corresponding parametrization that was previously given by Blümlein et. al. [9]. The authors came to the conclusion that *five* functions are necessary for a complete description. However, as demonstrated explicitely in appendix E of reference [10], one can show that one of their functions can actually be reexpressed in terms of the others, i.e. the structures given in their parametrization are not linearly independent if one takes into account the Dirac equation.

3.2 Reduction at the pion threshold

Within the soft pion kinematics described in section 2, all variables are required to be close to their values at the pion production threshold. Therefore, the *exact* threshold kinematics can serve as a reasonable approximation e.g. for the calculation of scattering amplitudes. For this purpose, let us consider the threshold case and provide some useful relations.

First, we recall that the invariant mass of the final nucleon-pion system is equal to

$$W_{\mathrm{th}}^2 = (M + m_\pi)^2. \tag{27}$$

This is equivalent to the statement that the four-momenta k and p' are proportional,

$$k = \frac{m_\pi}{M} p', \tag{28}$$

hence the number of independent invariants is reduced. In particular, we find that Δ^2 can be expressed through t and that α becomes a constant:

$$\Delta_{\mathrm{th}}^2 = \frac{M + m_\pi}{M} t + m_\pi^2, \qquad \alpha_{\mathrm{th}} = \frac{m_\pi}{M + m_\pi}. \tag{29}$$

As a further consequence of the relation (28), the Dirac structures involving e.g. $\Gamma_1 = \gamma_5$ and $\Gamma_3 = \not{k}\gamma_5/M$ are no longer linearly independent. Therefore, the number of πN distributions can be reduced by two. For example, in the case of the vector distributions H_i, we arrive at

$$\int \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{\text{ß}\lambda x \bar{p}\cdot n} \langle N'\pi^a | \bar{\psi}(-\lambda n/2) \not{n} \psi(\lambda n/2) | N \rangle = \frac{\text{ß} g_A}{M f_\pi} \bar{U}' \left[H_{1\mathrm{th}}^{(0)} + \frac{M \not{n}}{n\cdot\bar{p}} H_{2\mathrm{th}}^{(0)} \right] \gamma_5 \tau^a U \tag{30}$$

and

$$\begin{aligned} &\int \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{\text{ß}\lambda x \bar{p}\cdot n} \langle N'\pi^a | \bar{\psi}(-\lambda n/2) \not{n} \tau^b \psi(\lambda n/2) | N \rangle \\ &\quad = \frac{\text{ß} g_A}{M f_\pi} \bar{U}' \left[(\delta^{ab} H_{1\mathrm{th}}^{(+)} + \text{ß}\varepsilon^{abc}\tau^c H_{1\mathrm{th}}^{(-)}) + \frac{M\not{n}}{n\cdot\bar{p}} (\delta^{ab} H_{2\mathrm{th}}^{(+)} + \text{ß}\varepsilon^{abc}\tau^c H_{2\mathrm{th}}^{(-)}) \right] \gamma_5 U, \end{aligned} \tag{31}$$

with the threshold pion-nucleon distributions $H_{1\mathrm{th}}$ and $H_{2\mathrm{th}}$ given by

$$H_{1\mathrm{th}}^{(0,\pm)}(x,\xi,t) = H_1^{(0,\pm)}(x,\xi,\Delta_{\mathrm{th}}^2,\alpha_{\mathrm{th}},t,W_{\mathrm{th}}^2) + \frac{m_\pi}{M} H_3^{(0,\pm)}(x,\xi,\Delta_{\mathrm{th}}^2,\alpha_{\mathrm{th}},t,W_{\mathrm{th}}^2) \tag{32}$$

$$H_{2\mathrm{th}}^{(0,\pm)}(x,\xi,t) = H_2^{(0,\pm)}(x,\xi,\Delta_{\mathrm{th}}^2,\alpha_{\mathrm{th}},t,W_{\mathrm{th}}^2) + \frac{m_\pi}{M} H_4^{(0,\pm)}(x,\xi,\Delta_{\mathrm{th}}^2,\alpha_{\mathrm{th}},t,W_{\mathrm{th}}^2). \tag{33}$$

Analogously, we define $\tilde{H}_{1\mathrm{th}}^{(0,\pm)}$ and $\tilde{H}_{2\mathrm{th}}^{(0,\pm)}$ for the axial operators and $H_{1\mathrm{th}}^{(G)}$ and $H_{2\mathrm{th}}^{(G)}$ for the gluon operator.

3.3 Moments of the πN distributions

The moments of ordinary nucleon GPDs are polynomials in the skewedness variable, where the coefficients give the nucleon form factors of the corresponding local twist-2 operators [6, 11]. For example, taking the first moments of the quark GPDs yields

$$\int \mathrm{d}x\, H(x,\xi,\Delta^2) = F_1(\Delta^2), \qquad \int \mathrm{d}x\, E(x,\xi,\Delta^2) = F_2(\Delta^2), \tag{34}$$

$$\int \mathrm{d}x\, \tilde{H}(x,\xi,\Delta^2) = G_A(\Delta^2), \qquad \int \mathrm{d}x\, \tilde{E}(x,\xi,\Delta^2) = G_P(\Delta^2), \tag{35}$$

where F_1 and F_2 are the Dirac and Pauli form factor and G_A and G_P the axial and pseudoscalar form factor, respectively. Concerning the second moment, we have e.g.

$$\begin{aligned}
\int \mathrm{d}x\, x \left[H^{(S)}(x,\xi,\Delta^2) + \frac{1}{2} H^{(G)}(x,\xi,\Delta^2) \right] &= A(\Delta^2) + C(\Delta^2)\,(2\xi)^2, & (36)\\
\int \mathrm{d}x\, x \left[E^{(S)}(x,\xi,\Delta^2) + \frac{1}{2} E^{(G)}(x,\xi,\Delta^2) \right] &= B(\Delta^2) - C(\Delta^2)\,(2\xi)^2, & (37)
\end{aligned}$$

where A, B, and C are the form factors of the energy-momentum tensor. Similar polynomiality conditions hold for the moments of the πN distributions, their moments are polynomials in the variables ξ and α. We shall demonstrate this now explicitely for two examples.

3.3.1 First moment: The local limit

The first moments of the πN distributions $H_i^{(0,\pm)}$ are related to the form factors of the matrix element which describes pion emission from the nucleon induced by the the local vector current (the hadronic ingredient of the pion electroproduction amplitude). For our purposes, the following parametrization in terms of pion emission form factors A_i is convenient:

$$\begin{aligned}
&\langle N(p')\, \pi^a(k) | \bar{\psi} \gamma^\mu \left\{ \begin{array}{c} 1 \\ \tau^b \end{array} \right\} \psi | N(p) \rangle \\
&= \frac{\text{ß} g_A}{M f_\pi} \sum_{i=1}^{8} \bar{U}(p') \left\{ \begin{array}{c} \tau^a A_i^{(0)} \\ \delta^{ab} A^{(+)} + \text{ß} \varepsilon^{abc} \tau^c A^{(-)} \end{array} \right\} \Gamma_i^\mu U(p),
\end{aligned} \tag{38}$$

where the set of Dirac matrices is chosen as

$$\{\Gamma_1^\mu, \ldots, \Gamma_8^\mu\} = \{\bar{p}^\mu,\, \Delta^\mu,\, k^\mu,\, \gamma^\mu,\, \not{k}\bar{p}^\mu,\, \not{k}\Delta^\mu,\, \not{k}k^\mu,\, \not{k}\gamma^\mu\}\gamma_5. \tag{39}$$

(For a traditional parametrization we refer to Amaldi et. al. [12].) The form factors are functions of three independent invariants, e.g. Δ^2, W^2, and t. Current conservation reduces the number of independent form factors to six:

$$\begin{aligned}
(W^2 - M^2)A_1 + 2\Delta^2 A_2 + (W^2 + u - 2M^2)A_3 + 4MA_4 + 2(W^2 - M^2)A_8 &= 0 & (40)\\
2A_4 + (W^2 - M^2)A_5 + 2\Delta^2 A_6 + 2(W^2 + u - 2M^2)A_7 &= 0. & (41)
\end{aligned}$$

From the contraction of the matrix element (38) with the lightcone vector n, it follows that the first moments of the πN distributions are polynomials in $\Delta \cdot n / \bar{p} \cdot n = -2\xi$ and $k \cdot n / \bar{p} \cdot n = \alpha(1-\xi) \equiv \bar{\alpha}$:

$$\int_{-1}^{1} \mathrm{d}x\, H_1 = A_1 - 2\xi A_2 + \bar{\alpha} A_3, \qquad M \int_{-1}^{1} \mathrm{d}x\, H_2 = A_4, \tag{42}$$

$$\frac{1}{M}\int\limits_{-1}^{1} \mathrm{d}x\, H_3 = A_5 - 2\xi A_6 + \bar{\alpha} A_7, \qquad \int\limits_{-1}^{1} \mathrm{d}x\, H_4 = A_8. \tag{43}$$

Note that the current conservation relations (40) and (41) impose nontrivial conditions on the moments of the πN distributions.

3.3.2 Second moment: The energy-momentum tensor

The second moment of the πN distributions $H_i^{(0)}$ and $H_i^{(G)}$ is related to the form factors of the amplitude for pion emission induced by the energy-momentum tensor, which reads

$$\mathcal{T}^{\mu\nu} = \frac{\text{ß}}{2}\bar{\psi}\gamma^{\{\mu}(\overrightarrow{D} - \overleftarrow{D})^{\nu\}}\psi + \frac{g^{\mu\nu}}{4}F^{\rho\sigma}F_{\rho\sigma} + F^{\mu\rho}F_{\rho}{}^{\nu}, \tag{44}$$

where the curly brackets denote symmetrization of the indices and D the covariant derivative. We parametrize this amplitude as follows:

$$\langle N(p')\,\pi^a(k)|\mathcal{T}^{\mu\nu}|N_i(p)\rangle = \frac{\text{ß} g_A}{M f_\pi}\sum_{i=1}^{20}\bar{U}(p')\,\tau^a\Gamma_i^{\mu\nu}B_i U, \tag{45}$$

where the Dirac matrices are

$$\begin{aligned}\{\Gamma_1^{\mu\nu},\ldots,\Gamma_{20}^{\mu\nu}\} \;=\; & \{g^{\mu\nu},\, \bar{p}^\mu\bar{p}^\nu,\, \Delta^\mu\Delta^\nu,\, k^\mu k^\nu,\, \bar{p}^{\{\mu}\Delta^{\nu\}},\, \bar{p}^{\{\mu}k^{\nu\}},\, \Delta^{\{\mu}k^{\nu\}},\\ & g^{\mu\nu}\not{k},\, \bar{p}^\mu\bar{p}^\nu\not{k},\, \Delta^\mu\Delta^\nu\not{k},\, k^\mu k^\nu\not{k},\, \bar{p}^{\{\mu}\Delta^{\nu\}}\not{k},\, \bar{p}^{\{\mu}k^{\nu\}}\not{k},\, \Delta^{\{\mu}k^{\nu\}}\not{k},\\ & \gamma^{\{\mu}\bar{p}^{\nu\}},\, \gamma^{\{\mu}\,\Delta^{\nu\}},\, \gamma^{\{\mu}k^{\nu\}},\\ & \not{k}\gamma^{\{\mu},\, \not{k}\gamma^{\{\mu}\bar{p}^{\nu\}},\, \not{k}\gamma^{\{\mu}\Delta^{\nu\}},\, \not{k}\gamma^{\{\mu}k^{\nu\}}\}\gamma_5.\end{aligned} \tag{46}$$

As in the case of the vector current, the form factors B_i are functions of e.g. Δ^2, W^2, and t. From energy-momentum conservation, we have derived the following set of constraints:

$$\begin{aligned}& 4B_1 + 4\Delta^2 B_3 + (W^2 - M^2)B_5\\ &\quad + (W^2 + u - 2M^2)B_7 + 4MB_{16} + 2(W^2 - M^2)B_{19} = 0,\end{aligned} \tag{47}$$

$$2(W^2 - M^2)B_2 + 2\Delta^2 B_5 + (W^2 + u - 2M^2)B_6 + 4MB_{15} + 2(W^2 - M^2)B_{18} = 0, \tag{48}$$

$$4(W^2 + u - 2M^2)B_4 + (W^2 - M^2)B_6 + 2\Delta^2 B_7 + 4MB_{17} + 2(W^2 - M^2)B_{20} = 0, \tag{49}$$

$$4B_8 + 4\Delta^2 B_{10} + (W^2 - M^2)B_{12} + (W^2 + u - 2M^2)B_{14} + 2B_{16} = 0, \tag{50}$$

$$2(W^2 - M^2)B_9 + 2\Delta^2 B_{12} + (W^2 + u - 2M^2)B_{13} + 2B_{15} = 0, \tag{51}$$

$$2(W^2 + u - 2M^2)B_{11} + (W^2 - M^2)B_{13} + 2\Delta^2 B_{14} + 2B_{17} = 0, \tag{52}$$

$$(W^2 - M^2)B_{15} + 2\Delta^2 B_{16} + (W^2 + u - 2M^2)B_{17} = 0, \tag{53}$$

$$(W^2 - M^2)B_{18} + 2\Delta^2 B_{19} + (W^2 + u - 2M^2)B_{20} = 0, \tag{54}$$

which reduces the number of independent functions B_i to twelve. The polynomiality conditions read

$$\int_{-1}^{1} \mathrm{d}x\, x \left(H_1^{(0)} + \frac{1}{2} H_1^{(G)} \right) = B_2 + B_3(2\xi)^2 + B_4\bar{\alpha}^2 + B_5(-2\xi) + B_6\bar{\alpha} + B_7(-2\xi\bar{\alpha}), \tag{55}$$

$$M \int_{-1}^{1} \mathrm{d}x\, x \left(H_2^{(0)} + \frac{1}{2} H_2^{(G)} \right) = B_{15} + B_{16}(-2\xi) + B_{17}\bar{\alpha}, \tag{56}$$

$$\frac{1}{M} \int_{-1}^{1} \mathrm{d}x\, x \left(H_3^{(0)} + \frac{1}{2} H_3^{(G)} \right) = B_9 + B_{10}(2\xi)^2 + B_{11}\bar{\alpha}^2 + B_{12}(-2\xi) + B_{13}\bar{\alpha} + B_{14}(-2\xi\alpha), \tag{57}$$

$$\int_{-1}^{1} \mathrm{d}x\, x \left(H_4^{(0)} + \frac{1}{2} H_4^{(G)} \right) \;=\; B_{18} + B_{19}(-2\xi) + B_{20}\bar{\alpha}. \tag{58}$$

Thus we see that from current conservation and polynomiality it possible to uniquely determine the form factors B_i from the second moments of the πN distributions $H_i^{(0)} + H_i^{(G)}/2$.

4 Soft-pion theorems for pion emission from the nucleon induced by lightcone operators

In the last section, we have presented a parametrization of matrix elements for pion emission from the nucleon which is induced by twist-2 quark or gluon lightcone operators. Now, we turn to the calculation of these objects in the soft-pion region. For this, we rely on polology arguments, PCAC, and current algebra. The basic ideas are similar to those in the work of Guichon et. al. [8]. However, while Guichon et. al. excluded small momentum transfer, we shall consider this region in our derivation as well. Actually, for certain operators it was already shown in reference [13] that at small momentum transfer, the soft-pion theorems match the tree level results of a chiral perturbation theory treatment. This refutes opposite claim of ref. [14]. The results of [8] and [13] were confirmed also in ref. [15].

4.1 Simple pion emission from the nucleon

First, in order to introduce some notations and to demonstrate our approach on a very simple example, let us consider the emission of a pion from the nucleon, i.e. the amplitude $\mathcal{M}(N'\pi|N)$ defined through

$$\mathcal{M}(N(p')\,\pi^a(k)|N(p)) \equiv \lim_{k^2\to m_\pi^2} \frac{k^2-m_\pi^2}{ß} \langle N(p')|\Phi^a|N(p)\rangle, \tag{59}$$

where Φ^a is the interpolating pion field

$$\Phi^a \equiv \frac{\partial\cdot A^a}{f_\pi m_\pi^2}, \qquad \langle 0|\Phi^a|\pi^b\rangle = \delta^{ab}, \tag{60}$$

A^a_ν the axial current,

$$A^a_\nu = \bar\psi\gamma_\nu\gamma_5\frac{\tau^a}{2}\psi, \tag{61}$$

and (only within this subsection) k denotes the nucleon momentum difference,

$$k^\nu = (p-p')^\nu. \tag{62}$$

If we relax somewhat about the on-shell requirement for the pion, we can also write

$$\langle N(p)\,\pi^a(k)|N(p)\rangle = (2\pi)^4\delta(p'+k-p)\,\mathcal{M}(N(p')\,\pi^a(k)|N(p)). \tag{63}$$

To derive the soft-pion theorem for the amplitude $\mathcal{M}(N'\pi|N)$, we start from the nucleon matrix element

$$I_\nu = \frac{1}{f_\pi}\langle N(p')|A^a_\nu|N(p)\rangle. \tag{64}$$

Applying the definition of the interpolating pion field (60) yields the identity

$$k^\nu I_\nu = ß m_\pi^2\langle N(p')|\Phi^a|N(p)\rangle. \tag{65}$$

This equation is now investigated in the region where k is small,

$$k\sim m_\pi\sim\varepsilon$$

i.e. we search for the leading contributions in ε to both sides of equation (65).

The nucleon matrix element on the left hand side is parametrized in terms of the axial form factor G_A and the pseudoscalar form factor G_P,

$$I_\nu = \frac{1}{f_\pi}\langle N'|A^a_\nu|N\rangle = \frac{1}{f_\pi}\bar U'\left[G_A(k^2)\gamma_\nu - G_P(k^2)\frac{k_\nu}{2M}\right]\gamma_5\frac{\tau^a}{2}U. \tag{66}$$

For small k^2, G_A approaches the axial coupling constant and G_P is dominated by the corresponding pion pole contribution,

$$G_A(k^2) = g_A + \mathcal{O}(\varepsilon^2), \qquad G_P(k^2) = -\frac{(2M)^2 g_A}{k^2 - m_\pi^2} + \mathcal{O}(1/\varepsilon). \tag{67}$$

The insertion of these relations and the use of the Dirac equation yield

$$k^\nu I_\nu = -\frac{m_\pi^2}{k^2 - m_\pi^2} \bar{U}' \frac{g_A}{2 f_\pi} \not{k} \gamma_5 \tau^a U + \mathcal{O}(\varepsilon^2). \tag{68}$$

The right hand side of equation (65) has an explicit factor of $m_\pi^2 \sim \varepsilon^2$. This small factor can be compensated only by the pion pole contribution,

$$\text{ß} m_\pi^2 \langle N(p')|\Phi^a|N(p)\rangle = \text{ß} m_\pi^2 \frac{\text{ß}}{k^2 - m_\pi^2} \mathcal{M}(N(p')\pi^a(k)|N(p)) + \mathcal{O}(\varepsilon^2). \tag{69}$$

So finally, the equation (65) leads to

$$\mathcal{M}(N'\pi^a|N) = \bar{U}' \frac{g_A}{f_\pi} \not{k} \gamma_5 \frac{\tau^a}{2} U + \mathcal{O}(\varepsilon^2), \tag{70}$$

from which in turn the Goldberger-Treiman relation emerges, if we parametrize $\mathcal{M}(N'\pi|N)$ in terms of the pion-nucleon coupling constant $g_{\pi NN}$ as usual. The line of argument that lead to this well-known result is now generalized to the case when a lightcone operator is present.

4.2 Pion emission induced by lightcone operators

4.2.1 Isovector quark operator of vector type

Now we turn to the pion emission induced by lightcone operators. First, we deal with the isovector quark operator of vector type which we shall refer to as $O^b(\lambda)$ in the following:

$$O^b(\lambda) \equiv \bar{\psi}(-\lambda n/2) \not{n} \tau^b \psi(\lambda n/2). \tag{71}$$

We recall that in terms of local operators it reads

$$O^b(\lambda) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\lambda}{2}\right)^m \bar{\psi}(0) \, [n \cdot (\overrightarrow{\partial} - \overleftarrow{\partial})]^m \not{n} \tau^b \psi(0). \tag{72}$$

The derivation of the soft-pion formula for $\langle N'\pi|O^b(\lambda)|N\rangle$ starts from the following object denoted I_ν:

$$I_\nu \equiv \frac{1}{f_\pi} \int \mathrm{d}^4 z \, \mathrm{e}^{\text{ß} k \cdot z} \langle N(p')|T[A_\nu^a(z) \, O^b(\lambda)]|N(p)\rangle, \tag{73}$$

where T stands for the time-ordering prescription

$$T[A^a_\nu(z)\, O^b(\lambda)] = \theta(z_0)\, A^a_\nu(z)\, O^b(\lambda) + \theta(-z_0)\, O^b(\lambda)\, A^a_\nu(z) \tag{74}$$

Let us investigate the behavior of the matrix element I_ν in the soft-pion region

$$k \sim m_\pi \sim \varepsilon. \tag{75}$$

In this region, several momenta are close to the pion or nucleon mass shell:

$$k^2 = m_\pi^2 + \mathcal{O}(\varepsilon^2), \qquad W^2 = (p'+k)^2 = M^2 + \mathcal{O}(\varepsilon), \qquad u = (p-k)^2 = M^2 + \mathcal{O}(\varepsilon). \tag{76}$$

Generally, for a Green function of n Operators,

$$G_{1\ldots n}(\beta|\alpha) \equiv \int \mathrm{d}^4 z_1 \ldots \mathrm{d}^4 z_{n-1} \mathrm{e}^{\text{ß}(p_1\cdot z_1+\ldots+p_{n-1}\cdot z_{n-1})} \langle\beta|T[O_1(z_1)\ldots O_{n-1}(z_{n-1})\, O_n(0)]|\alpha\rangle, \tag{77}$$

we expect the following behavior when $|\alpha\rangle = |\beta\rangle = |0\rangle$ and the momentum $p \equiv p_1 + \ldots + p_r$ reaches the mass shell of a particle with mass m and quantum numbers s:

$$G_{1\ldots n}(0|0) \overset{(p_1+\ldots+p_r)^2=p^2\to m^2}{\longrightarrow} \sum_s G_{1\ldots r}(0|p,s) \frac{\text{ß}}{p^2-m^2} G_{r+1\ldots n}(p,s|0) \tag{78}$$

(see, e.g., Chapter 10.2 in the textbook of Weinberg [16]). The matrix element I_ν can be related to a Green function of the type $G_{1\ldots 4}(0|0)$ according to the usual LSZ formalism. However, in our case we deal with the additional problem that *several* momentum combinations become nearly on-shell *at once*, as shown in the equations (76). We proceed by adding up these different pole contributions, each of which is of the type (78), and subtract appropriate terms where necessary to avoid a double counting. In this way we obtain the preliminary statement

$$I^\nu = I^\nu_{\pi(k)} + I^\nu_{N(p'+k)} + I^\nu_{N(p-k)} + \mathcal{O}(\varepsilon^0), \qquad \text{(preliminary)} \tag{79}$$

where $I^\nu_{\pi(k)}$ denotes the pion pole in the variable k^2,

$$I^\nu \overset{k^2\to m_\pi^2}{\longrightarrow} \frac{1}{f_\pi} \langle 0|A^{\nu a}(0)|\pi^c(k)\rangle \frac{\text{ß}}{k^2-m_\pi^2} \langle N(p')\,\pi^c(k)|O^b(\lambda)|N(p)\rangle \equiv I^\nu_{\pi(k)} = \mathcal{O}(1/\varepsilon), \tag{80}$$

$I^\nu_{N(p'+k)}$ the nucleon pole in the variable W^2 with pion-pole subtraction,

$$\begin{aligned} I^\nu - I^\nu_{\pi(k)} \quad \overset{W^2\to M^2}{\longrightarrow} \quad & \frac{1}{f_\pi}\Big[\langle N(p')|A^{\nu a}|N_\sigma(p'+k)\rangle \\ & -\langle 0|A^{\nu a}|\pi^c(k)\rangle \frac{\text{ß}}{k^2-m_\pi^2} \mathcal{M}(N(p')\pi^c(k)|N_\sigma(p'+k))\Big] && (81) \\ & \times \frac{\text{ß}}{W^2-M^2} \langle N_\sigma(p'+k)|O^b(\lambda)|N(p)\rangle \equiv I^\nu_{N(p'+k)} = \mathcal{O}(1/\varepsilon), && (82) \end{aligned}$$

and $I^\nu_{N(p-k)}$ the nucleon pole in u with pion-pole subtraction,

$$\begin{aligned} I^\nu - I^\nu_{\pi(k)} \quad &\stackrel{u\to M^2}{\longrightarrow} \quad \frac{1}{f_\pi}\langle N(p')|O^b(\lambda)|N_\sigma(p-k)\rangle \frac{\text{ß}}{u-M^2}\Big[\langle N_\sigma(p-k)|A^{\nu a}(0)|N(p)\rangle \\ &\qquad -\langle 0|A^{\nu a}|\pi^c(k)\rangle \frac{\text{ß}}{k^2-m_\pi^2}\mathcal{M}(N_\sigma(p-k)\,\pi^c(k)|N(p))\Big] \qquad (83) \\ &\qquad \equiv I^\nu_{N(p-k)} = \mathcal{O}(1/\varepsilon). \qquad (84) \end{aligned}$$

In the previous formulas, c denotes the pion's isospin and σ a combined nucleon spin-isospin index. Summation over quantum numbers of intermediate states, such as c and σ here, are henceforth understood.

An additional pole that we have not mentioned yet has to be considered in the kinematics of small momentum transfer, $t \sim \varepsilon^2$, since then

$$t = m_\pi^2 + \mathcal{O}(\varepsilon^2). \qquad (85)$$

It has the form

$$\begin{aligned} I^\nu - I^\nu_{\pi(k)} \quad &\stackrel{t\to m_\pi^2}{\longrightarrow} \quad \frac{1}{f_\pi}\Big[\int \mathrm{d}^4z\, \mathrm{e}^{\text{ß}k\cdot z}\langle 0|T[A^{\nu a}(z)\,O^b]|\pi^d(p-p')\rangle \\ &\qquad -\langle 0|A^{\nu a}|\pi^c(k)\rangle \frac{\text{ß}}{k^2-m_\pi^2}\langle \pi^c(k)|O^b|\pi^d(p-p')\rangle\Big] \qquad (86) \\ &\qquad \times\frac{\text{ß}}{t-m_\pi^2}\mathcal{M}(N'\pi^d(p-p')|N) \equiv I^\nu_{\pi(p-p')}, \qquad (87) \end{aligned}$$

where a $\pi(k)$-pole subtraction has been performed. Therefore, to cover the whole region of momentum transfer $-t \lesssim M^2$, we modify equation (79) by adding this contribution,

$$I^\nu = I^\nu_{\pi(k)} + I^\nu_{N(p'+k)} + I^\nu_{N(p-k)} + I^\nu_{\pi(p-p')} + \mathcal{O}(\varepsilon^0), \qquad (88)$$

always keeping in mind that $I^\nu_{\pi(p-p')}$ is suppressed automatically as $-t$ grows:

$$I^\nu_{\pi(p-p')} = \begin{cases} \mathcal{O}(1/\varepsilon) & t \sim \varepsilon^2 \\ \mathcal{O}(\varepsilon^0) & -t \gg \varepsilon^2 \end{cases}. \qquad (89)$$

This closes the discussion about the pole structure of the matrix element I^ν.

Next we turn to a simple identity for I_ν that follows when we use the definition of the interpolating pion field (60):

$$k^\nu I_\nu = \text{ß}\int \mathrm{d}^4z\, \mathrm{e}^{\text{ß}k\cdot z}\left\{\frac{\delta(z_0)}{f_\pi}\langle N'|[A_0^a(z), O^b(\lambda)]|N\rangle + m_\pi^2\langle N'|T[\Phi^a(z)\,O^b(\lambda)]|N\rangle\right\}. \qquad (90)$$

Let us exploit this identity in the soft-pion region (75). According to our previous demonstrations, the leading contributions to the left hand side are simply obtained by contracting k with the poles in (88). The contraction with the $\pi(k)$ pole yields

$$k_\nu I^\nu_{\pi(k)} = -\frac{k^2}{k^2-m_\pi^2}\langle N'\pi^a(k)|O^b(\lambda)|N\rangle + \mathcal{O}(\varepsilon), \qquad (91)$$

where we have used the definition of the pion decay constant,

$$\langle 0|A_\nu^a|\pi^b(k)\rangle = ß f_\pi k_\nu \delta^{ab}. \tag{92}$$

For the sum of the nucleon poles we obtain

$$\begin{aligned} k_\nu(I^\nu_{N(p'+k)} + I^\nu_{N(p-k)}) &= \bar{U}'\frac{g_A}{2f_\pi}\not{k}\gamma_5\tau^a\frac{ß(\not{p}'+M)}{2p'\cdot k}\Gamma^{(V)}(p'+k,p,\lambda)\,\tau^b U \\ &\quad +\bar{U}'\Gamma^{(V)}(p',p-k,\lambda)\,\tau^b\frac{ß(\not{p}+M)}{-2p\cdot k}\frac{g_A}{2f_\pi}\not{k}\gamma_5\tau^a U + \mathcal{O}(\varepsilon), \end{aligned} \tag{93}$$

where we have applied

$$\frac{k^\nu}{f_\pi}\langle N(p_2)|A_\nu^a|N(p_1+k)\rangle = -\frac{m_\pi^2}{k^2-m_\pi^2}\bar{U}(p_2)\frac{g_A}{2f_\pi}\not{k}\gamma_5\tau^a U(p_1) + \mathcal{O}(\varepsilon^2) \tag{94}$$

and

$$\mathcal{M}(N(p_2)\,\pi^a(k)|N(p_1)) = \bar{U}(p_2)\frac{g_A}{2f_\pi}\not{k}\gamma_5\tau^a U(p_1) + \mathcal{O}(\varepsilon^2), \tag{95}$$

as discussed in the previous subsection. Moreover, we have introduced the parametrization

$$\langle N(p_2)|O^b(\lambda)|N(p_1)\rangle = \bar{U}(p_2)\,\Gamma^{(V)}(p_2,p_1,\lambda)\,\tau^b\,U(p_1), \tag{96}$$

where in terms of generalized parton distributions one usually defines

$$\Gamma^{(V)}(p_2,p_1,\lambda) \equiv \int_{-1}^{1} \mathrm{d}x\,\mathrm{e}^{-ß\lambda x(p_2+p_1)\cdot n/2}\left[H^{(V)}\not{n} + E^{(V)}\frac{ß\sigma(n,p_2-p_1)}{2M}\right], \tag{97}$$

with

$$H^{(V)} = H^{(V)}\left(x, \frac{(p_1-p_2)\cdot n}{(p_1+p_2)\cdot n}, (p_1-p_2)^2\right) \tag{98}$$

and analogously for $E^{(V)}$. Further, we mention that it is useful to rewrite the contraction with the $\pi(p-p')$ pole in the form

$$\begin{aligned} &k_\nu I^\nu_{\pi(p-p')} \\ &= \left[\frac{ß}{f_\pi}\int \mathrm{d}^4z\,\mathrm{e}^{ßk\cdot z}\{\delta(z_0)\langle 0|[A_0^a(z),O^b]|\pi^d(p-p')\rangle + m_\pi^2\langle 0|T[\Phi^a(z)\,O^b]|\pi^d(p-p')\rangle\}\right. \\ &\quad \left. -\frac{k_\nu}{f_\pi}\langle 0|A^{\nu a}|\pi^c(k)\rangle\frac{ß}{k^2-m_\pi^2}\langle\pi^c(k)|O^b|\pi^d(p-p')\rangle\right]\frac{ß}{t-m_\pi^2}\mathcal{M}(N'\,\pi^d(p-p')|N) \quad (99) \\ &= \left[\frac{ß}{f_\pi}\langle 0|[Q_5^a(0),O^b]|\pi^d(p-p')\rangle + ßm_\pi^2\int \mathrm{d}^4z\,\mathrm{e}^{ßk\cdot z}\langle 0|T[\Phi^a(z)\,O^b]|\pi^d(p-p')\rangle\right. \\ &\quad \left. +\frac{k^2}{k^2-m_\pi^2}\langle\pi^a(k)|O^b|\pi^d(p-p')\rangle\right]\frac{ß}{t-m_\pi^2}\bar{U}'\frac{g_A}{2f_\pi}(\not{p}-\not{p}')\gamma_5\tau^d U + \mathcal{O}(\varepsilon), \quad (100) \end{aligned}$$

where Q_5^a is the axial charge,

$$Q_5^a(z_0) \equiv \int \mathrm{d}^3 z\, A_0^a(z). \tag{101}$$

So much for the left hand side of the identity (90). Now, what are the main terms on the right hand side?

First, we turn to the commutator term. It can be approximated by neglecting the soft momentum k,

$$\text{ß} \int \mathrm{d}^4 z\, \mathrm{e}^{\text{ß} k\cdot z} \frac{\delta(z_0)}{f_\pi} \langle N'|[A_0^a(z), O^b(\lambda)]|N\rangle = \frac{\text{ß}}{f_\pi} \langle N'|[Q_5^a(0), O^b(\lambda)]|N\rangle + \mathcal{O}(\varepsilon). \tag{102}$$

For the calculation of the commutator $[Q_5^a(0), O^b(\lambda)]$, we insert $O^b(\lambda)$ in the form (72). Then we have to deal with objects of the type $[Q_5^a(0), (n\cdot\partial)^m \psi(0)]$ and $[Q_5^a(0), (n\cdot\partial)^m \bar{\psi}(0)]$. For $m = 0$, these commutators are well-known from the transformation of the fields under the axial part of the chiral rotation:

$$[Q_5^a(0),\, \psi(0)] = -\gamma_5 \frac{\tau^a}{2} \psi(0), \qquad [Q_5^a(0), \bar{\psi}(0)] = -\bar{\psi}(0)\gamma_5 \frac{\tau^a}{2}. \tag{103}$$

In the general case $m \geq 0$, we can rewrite e.g.

$$[Q_5^a, (n\cdot\partial)^m \psi] = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) (-1)^k (n\cdot\partial)^{m-k} [(n_0\partial_0)^k Q_5^a, \psi]. \tag{104}$$

The derivative of the axial charge is given as

$$\partial_0 Q_5^a(z_0) = \int \mathrm{d}^3 z\, \bar{\psi}(z)\, \gamma_5 \{\tau^a, \hat{m}\} \psi(z), \tag{105}$$

where $\hat{m} = \mathrm{diag}(m_u, m_d)$ is the quark mass matrix. According to the Gell-Mann-Oakes-Renner relation we have

$$m_u, m_d \propto m_\pi^2 \sim \varepsilon^2,$$

hence we can neglect the terms that involve derivatives of Q_5^a:

$$[Q_5^a, (n\cdot\partial)^m \psi] = (n\cdot\partial)^m [Q_5^a, \psi] + \mathcal{O}(\varepsilon^2) \tag{106}$$

and similar for $[Q_5^a, (n\cdot\partial)^m \bar{\psi}]$. In this way we obtain finally

$$[Q_5^a(0), O^b(\lambda)] = \text{ß}\varepsilon^{abc} O_5^c(\lambda) + \mathcal{O}(\varepsilon^2) \tag{107}$$

with

$$O_5^c(\lambda) \equiv \bar{\psi}(-\lambda n/2)\, \not{n} \gamma_5 \tau^c \psi(\lambda n/2). \tag{108}$$

Next, we discuss the second term on the right hand side of (90). It is accompanied by a small factor $m_\pi^2 \sim \varepsilon^2$, so again we focus on the pole contributions which are of order $1/\varepsilon^2$.

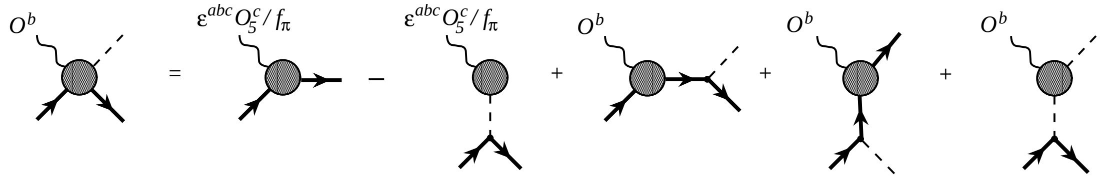

Figure 2: Illustration of the soft-pion theorem for the matrix element $\langle N(p')\,\pi^a(k)|O^b(\lambda)|N(p)\rangle$, where $O^b(\lambda)$ is the lightcone operator $\bar\psi(-\lambda n/2)\,\not{n}\tau^b\psi(\lambda n/2)$. The blobs denote insertion of the indicated operator while the pointlike vertices represent the standard pseudovector pion-nucleon coupling.

These are the pion pole in k^2 and, for small momentum transfer, the pion pole in t. Again taking into account a double-pole subtraction, we thus get

$$
\begin{aligned}
&ßm_\pi^2 \int \mathrm{d}^4 z\, \mathrm{e}^{ßk\cdot z} \langle N'|T[\Phi^a(z)\, O^b]|N\rangle \\
&= \; -\frac{m_\pi^2}{k^2 - m_\pi} \langle N'\pi^a(k)|O^b|N\rangle + \left\{ ßm_\pi^2 \int \mathrm{d}^4 z\, \mathrm{e}^{ßk\cdot z} \langle 0|T[\Phi^a(z)\, O^b]|\pi^d(p-p')\rangle \right. \\
&\quad \left. + \frac{m_\pi^2}{k^2 - m_\pi^2} \langle \pi^a(k)|O^b(\lambda)|\pi^d(p-p')\rangle \right\} \frac{ß}{t - m_\pi^2} \mathcal{M}(N'\pi^d(p-p')|N) + \mathcal{O}(\varepsilon). \quad (109)
\end{aligned}
$$

Now we can collect all approximations and insert them into (90) (note that the terms which involve the time-orderd product $T[\Phi^a(z), O^b]$ cancel). Then we can solve the equation for the matrix element $\langle N'\pi^a|O^b|N\rangle$ and arrive at the soft-pion theorem

$$
\begin{aligned}
&\langle N'\pi^a(k)|O^b(\lambda)|N\rangle \\
&= \; \frac{\varepsilon^{abc}}{f_\pi} \left[\langle N'|O_5^c(\lambda)|N\rangle - \langle 0|O_5^c(\lambda)|\pi^d(p-p')\rangle \frac{ß}{t - m_\pi^2} \bar U' \frac{g_A}{2f_\pi} (\not{p} - \not{p}')\gamma_5 \tau^d U \right] \\
&\quad + \frac{ßg_A}{2f_\pi} \bar U' \left[\not{k}\gamma_5\tau^a \frac{\not{p}' + M}{2p'\cdot k} \Gamma^{(V)}(p'+k, p, \lambda)\,\tau^b + \Gamma^{(V)}(p', p-k, \lambda)\,\tau^b \frac{\not{p} + M}{-2p\cdot k} \not{k}\gamma_5\tau^a \right] U \\
&\quad + \langle \pi^a(k)|O^b(\lambda)|\pi^d(p-p')\rangle \frac{ß}{t - m_\pi^2} \bar U' \frac{g_A}{2f_\pi} (\not{p} - \not{p}')\gamma_5\tau^d U + \mathcal{O}(\varepsilon), \quad (110)
\end{aligned}
$$

see also figure 2. The pion matrix elements in this formula are parametrized as

$$
\langle 0|O_5^c(\lambda)|\pi^d(p-p')\rangle = 2ßf_\pi(p-p')\cdot n\,\delta^{cd} \int_{-1}^{1} \mathrm{d}x\, \mathrm{e}^{-ß\lambda x(p-p')\cdot n/2} \phi_\pi(x),
$$

where ϕ_π is the pion distribution amplitude, and

$$
\begin{aligned}
&\langle \pi^a(k)|O^b(\lambda)|\pi^d(p-p')\rangle \\
&\quad = ß\varepsilon^{abd}(k+p-p')\cdot n \int_{-1}^{1} \mathrm{d}x\, \mathrm{e}^{-ß\lambda x(k+p-p')\cdot n/2} H_\pi^{(V)}\left(x, \frac{(p-p'-k)\cdot n}{(p-p'+k)\cdot n}, 0\right) + \mathcal{O}(\varepsilon^3),
\end{aligned}
$$

In the last line, we have neglected the momentum transfer in the argument of the pion GPD $H_\pi^{(V)}$, because the $\pi(p-p')$ pole terms only contribute in the region where $t \sim \Delta^2 \sim \varepsilon^2$. For a parametrization of the nucleon matrix element of $O_5^b(\lambda)$ in terms of GPDs see below, formulas (132) and (133).

For reasons of consistency, it is advisable to check that the $\pi(p-p')$ pole contributions in the soft-pion formula is really negligible for $-t \gg \varepsilon^2$, as we have promised at the beginning of the calculation. In this region of moderate momentum transfer, we have to distinguish two cases which are both kinematically allowed:

$$(p-p')\cdot n \sim k\cdot n \sim \varepsilon \qquad \text{or} \qquad (p-p')\cdot n \gg k\cdot n \sim \varepsilon. \tag{111}$$

In the first case, both pion pole terms are individually suppressed because the pion matrix elements are proportional to combinations of $(p-p')\cdot n$ and $k\cdot n$. In the latter case, the *combination* of both terms is small because

$$\begin{aligned} & (k+p-p')\cdot n \int\limits_{-1}^{1} \mathrm{d}x\, \mathrm{e}^{-\text{ß}\lambda x(k+p-p')\cdot n/2} H_\pi^{(V)}\left(x, \frac{(p-p'-k)\cdot n}{(p-p'+k)\cdot n}, 0\right) \\ \stackrel{(p-p')\cdot n \gg k\cdot n \sim \varepsilon}{=} \; & (p-p')\cdot n \int\limits_{-1}^{1} \mathrm{d}x\, \mathrm{e}^{-\text{ß}\lambda x(p-p')\cdot n/2} H_\pi^{(V)}(x,1,0) + \mathcal{O}(\varepsilon) \end{aligned} \tag{112}$$

and a soft-pion theorem for the pion GPD [17] reads

$$H_\pi^{(V)}(x,1,0) = 2\phi_\pi(x). \tag{113}$$

So we have confirmed that in the region of moderate momentum transfer the $\pi(p-p')$ pole is negligible. Therefore, our result coincides with that one of Guichon et. al. [8], who restricted themselves to the region $-t \gg \varepsilon^2$.

4.2.2 Isocalar quark operator of vector type and gluon operator

Soft-pion theorems similar to the previous one can also be derived for the isoscalar quark operator and the gluon operator. For the isoscalar quark operator, we get

$$\begin{aligned} & \langle N'\pi^a(k)|\bar\psi(-\lambda n/2)\not{n}\psi(\lambda n/2)|N\rangle \\ = \; & \frac{\text{ß} g_A}{2f_\pi}\bar U'\left[\not{k}\gamma_5\tau^a \frac{\not{p}'+M}{2p'\cdot k}\Gamma^{(S)}(p'+k,p,\lambda) - \Gamma^{(S)}(p',p-k,\lambda)\frac{\not{p}+M}{2p\cdot k}\not{k}\gamma_5\tau^a\right] U \\ & +\langle \pi^a(k)|\bar\psi(-\lambda n/2)\not{n}\psi(\lambda n/2)|\pi^c(p-p')\rangle \frac{\text{ß}}{t-m_\pi^2}\bar U' \frac{g_A}{2f_\pi}(\not{p}-\not{p}')\gamma_5\tau^c U + \mathcal{O}(\varepsilon), \end{aligned} \tag{114}$$

where $\Gamma^{(S)}$ can be obtained from the definition of $\Gamma^{(V)}$ in equation (97) by replacing the isovector GPDs $H^{(V)}$ and $E^{(V)}$ with the isoscalar ones, $H^{(S)}$ and $E^{(S)}$. For the gluon operator, we obtain

$$\begin{aligned}
&\langle N'\pi^a(k)|n_\mu F^{\mu\rho}(-\lambda n/2)\, F_{\rho\nu}(\lambda n/2)\, n^\nu|N\rangle \\
&= \quad \frac{ßg_A}{2f_\pi}\bar{U}'\left[\not{k}\gamma_5\tau^a\frac{\not{p}'+M}{2p'\cdot k}\Gamma^{(G)}(p'+k,p,\lambda)-\Gamma^{(G)}(p',p-k,\lambda)\frac{\not{p}+M}{2p\cdot k}\not{k}\gamma_5\tau^a\right]U \\
&\quad +\langle\pi^a(k)|n_\mu F^{\mu\rho}(-\lambda n/2)\, F_{\rho\nu}(\lambda n/2)\, n^\nu|\pi^c(p-p')\rangle\frac{ß}{t-m_\pi^2}\bar{U}'\frac{g_A}{2f_\pi}(\not{p}-\not{p}')\gamma_5\tau^c U \\
&\quad +\mathcal{O}(\varepsilon),
\end{aligned} \tag{115}$$

where

$$\Gamma^{(G)}(p_2,p_1,\lambda)=\frac{n\cdot(p_1+p_2)}{4}\int_{-1}^{1}\mathrm{d}x\,\mathrm{e}^{-ß\lambda x\,n\cdot(p_1+p_2)/2}x\left[H^{(G)}\not{n}+E^{(G)}\frac{ß\sigma(n,p_2-p_1)}{2M}\right], \tag{116}$$

with arguments of the gluon GPDs $H^{(G)}$ and $E^{(G)}$ as in (98). Because of the isoscalar nature of the two operators considered here, commutator terms do not appear in the results. Moreover, we remark that the discussion of the pion pole terms at moderate momentum transfer which we presented in the isovector case, can be repeated here in a similar way using the following soft-pion theorems for the gluon and the isoscalar pion GPD [17]:

$$H^{(S)}(x,1,0)=0, \qquad H^{(G)}(x,1,0)=0. \tag{117}$$

So again, for $-t\gg\varepsilon^2$ the pion pole contributions to the soft-pion theorem are negligible.

4.2.3 Isovector operator of axial vector type

Next we turn to the derivation of the soft-pion theorem for the matrix element of the operator

$$O_5^b(\lambda)=\bar{\psi}(-\lambda n/2)\not{n}\gamma_5\tau^b\psi(\lambda n/2). \tag{118}$$

For this purpose, we define

$$I^\nu\equiv\frac{1}{f_\pi}\int\mathrm{d}^4z\,\mathrm{e}^{ßk\cdot z}\langle N(p')|T[A_\nu^a(z)\,O_5^b(\lambda)]|N(p)\rangle \tag{119}$$

and obtain the identity

$$k_\nu I^\nu=ß\int\mathrm{d}^4z\,\mathrm{e}^{ßk\cdot z}\left\{\frac{\delta(z_0)}{f_\pi}\langle N'|[A_0^a(z),O_5^b(\lambda)]|N\rangle+m_\pi^2\langle N'|T[\Phi^a(z)\,O_5^b(\lambda)]|N\rangle\right\}. \tag{120}$$

Again, we make soft-pion expansions of both sides when $k\sim m_\pi\sim\varepsilon$. At this, we have to consider pion poles in k^2 and $\Delta^2=(p'+k-p)^2$ and nucleon poles:

$$I^\nu=I^\nu_{\pi(k)}+I^\nu_{\pi(\Delta)}+I^\nu_{N(p'+k)}+I^\nu_{N(p-k)}+\mathcal{O}(\varepsilon^0). \tag{121}$$

The $\pi(k)$ pole is given as

$$
\begin{aligned}
I^\nu \stackrel{k^2\to m_\pi^2}{\longrightarrow} &= \frac{1}{f_\pi}\langle 0|A^{\nu a}(0)|\pi^c(k)\rangle \frac{\text{ß}}{k^2-m_\pi^2}\langle N'\pi^c(k)|O_5^b(\lambda)|N\rangle && (122)\\
&= \frac{-k^\nu}{k^2-m_\pi^2}\langle N'\pi^a(k)|O_5^b(\lambda)|N\rangle \equiv I^\nu_{\pi(k)} && (123)
\end{aligned}
$$

For the $\pi(\Delta)$ pole we get

$$
\begin{aligned}
I^\nu - I^\nu_{\pi(k)} \stackrel{\Delta^2\to m_\pi^2}{\longrightarrow} &= \left[\frac{1}{f_\pi}\langle N'|A^{\nu a}(0)|N\pi^c(\Delta)\rangle + \frac{k^\nu}{k^2-m_\pi^2}\mathcal{M}(N'\pi^a(k)|N\pi^c(\Delta))\right]\\
&\times \frac{\text{ß}}{\Delta^2-m_\pi^2}\langle \pi^c(\Delta)|O^b(\lambda)|0\rangle \equiv I^\nu_{\pi(\Delta)}, && (124)
\end{aligned}
$$

where $\mathcal{M}(N'\pi^d(k)|N\pi^c(\Delta))$ denotes the pion-nucleon scattering amplitude,

$$
\langle N'\pi^d(k)|N\pi^c(\Delta)\rangle = (2\pi)^4\delta(p'+k-p-\Delta)\,\mathcal{M}(N'\pi^d(k)|N\pi^c(\Delta)). \qquad (125)
$$

This pion pole is non-negligible only if $-\Delta^2 \lesssim m_\pi^2 \sim \varepsilon^2$. The nucleon pole terms are

$$
\begin{aligned}
& I^\nu - I^\nu_{\pi(k)} - I^\nu_{\pi(\Delta)} && (126)\\
& \stackrel{W^2\to M^2}{\longrightarrow} \left[\frac{1}{f_\pi}\langle N'|A^{\nu a}|N_\sigma(p'+k)\rangle + \frac{k^\nu}{k^2-m_\pi^2}\mathcal{M}(N'\pi^a(k)|N_\sigma(p'+k))\right]\frac{\text{ß}}{W^2-M^2}\\
& \quad \times\left[\langle N_\sigma(p'+k)|O_5^b|N\rangle - \mathcal{M}(N_\sigma(p'+k)|N\pi^c(\Delta))\frac{\text{ß}}{\Delta^2-m_\pi^2}\langle\pi^c(\Delta)|O_5^b|0\rangle\right] && (127)\\
&= \bar{U}'\frac{\text{ß}g_A}{2f_\pi}\gamma^\nu\gamma_5\tau^a\frac{\not p'+M}{2p'\cdot k}\left[\Gamma_5^{(V)}(p'+k,p,\lambda)\,\tau^b + \frac{\text{ß}g_A}{2f_\pi}\not\Delta\gamma_5\tau^c\frac{\langle\pi^c(\Delta)|O_5^b|0\rangle}{\Delta^2-m_\pi^2}\right]U\\
& \quad +\mathcal{O}(\varepsilon^0) \equiv I^\nu_{N(p'+k)} && (128)
\end{aligned}
$$

and

$$
\begin{aligned}
& I^\nu - I^\nu_{\pi(k)} - I^\nu_{\pi(\Delta)} && (129)\\
& \stackrel{u\to M^2}{\longrightarrow} \left[\langle N'|O_5^b|N_\sigma(p-k)\rangle - \mathcal{M}(N'|N_\sigma(p-k)\pi^c(\Delta))\frac{\text{ß}}{\Delta^2-m_\pi^2}\langle\pi^c(\Delta)|O_5^b|0\rangle\right]\\
& \quad \times\frac{\text{ß}}{u-M^2}\left[\frac{1}{f_\pi}\langle N_\sigma(p-k)|A^{\nu a}|N\rangle + \frac{k^\nu}{k^2-m_\pi^2}\mathcal{M}(N_\sigma(p-k)\pi^a(k)|N)\right] && (130)\\
&= \bar{U}'\left[\Gamma_5^{(V)}(p',p-k)\,\tau^b + \frac{\text{ß}g_A}{2f_\pi}\not\Delta\gamma_5\tau^c\frac{\langle\pi^c(\Delta)|O_5^b|0\rangle}{\Delta^2-m_\pi^2}\right]\frac{\not p+M}{-2p\cdot k}\frac{\text{ß}g_A}{2f_\pi}\gamma^\nu\gamma_5\tau^a U\\
& \quad +\mathcal{O}(\varepsilon^0) \equiv I^\nu_{N(p-k)}, && (131)
\end{aligned}
$$

where we make use of the parametrization

$$
\langle N(p_2)|O_5^b(\lambda)|N(p_1)\rangle = \bar{U}(p_2)\,\Gamma_5^{(V)}(p_2,p_1,\lambda)\,\tau^b U(p_1) \qquad (132)
$$

with

$$\Gamma_5^{(V)}(p_2,p_1,\lambda) = \int\limits_{-1}^{1} \mathrm{d}x\, \mathrm{e}^{-ß\lambda x (p_1+p_2)\cdot n/2} \left[\tilde{H}^{(V)} \not{n} + \tilde{E}^{(V)} \frac{(p_2-p_1)\cdot n}{(2M)^2} (\not{p}_2 - \not{p}_1) \right] \gamma_5. \tag{133}$$

Note that in contrast to the previous case of the vector type operator $O^b(\lambda)$, a $\pi(\Delta)$ pole subtraction has been necessary to obtain the correct nucleon pole terms (128) and (131) without double counting.

From the soft-pion expansion for I^ν that we have determined now, we immediately obtain the left hand side of the identity (120), $k_\nu I^\nu$. The right hand side gives

$$\begin{aligned} & ß \int \mathrm{d}^4 z\, \mathrm{e}^{ßk\cdot z} \left\{ \frac{\delta(z_0)}{f_\pi} \langle N' | [A_0^a(z), O_5^b(\lambda)] | N\rangle + m_\pi^2 \langle N' | T[\Phi^a(z)\, O_5^b(\lambda)] | N \rangle \right\} \\ = \; & -\frac{\varepsilon^{abc}}{f_\pi} \langle N' | O^c(\lambda) | N\rangle - \frac{m_\pi^2}{k^2 - m_\pi^2} \langle N' \pi^a(k) | O_5^b(\lambda) | N \rangle \\ & + \left[\frac{k^\nu}{f_\pi} \langle N' | A_\nu^a | N \pi^c(\Delta)\rangle + \frac{m_\pi^2}{k^2 - m_\pi^2} \mathcal{M}(N'\pi^a(k)|N\pi^c(\Delta)) \right] \frac{ß}{\Delta^2 - m_\pi^2} \langle \pi^c(\Delta) | O_5^b(\lambda) | 0\rangle \\ & + \mathcal{O}(\varepsilon). \end{aligned} \tag{134}$$

In this way, we obtain from (120) the following first version of the soft-pion theorem:

$$\begin{aligned} & \langle N' \pi^a(k) | O_5^b(\lambda) | N\rangle \\ = \; & \frac{\varepsilon^{abc}}{f_\pi} \langle N' | O^c(\lambda) | N \rangle + \mathcal{M}(N'\pi^a(k)|N\pi^c(\Delta)) \frac{ß}{\Delta^2 - m_\pi^2} \langle \pi^c(\Delta) | O_5^b(\lambda) | 0 \rangle \\ & + \bar{U}' \frac{ß g_A}{2 f_\pi} \not{k} \gamma_5 \tau^a \frac{\not{p}' + M}{2p'\cdot k} \left[\Gamma_5^{(V)}(p'+k, p, \lambda)\, \tau^b + \frac{ß g_A}{2 f_\pi} \not{\Delta} \gamma_5 \tau^c \frac{\langle \pi^c(\Delta) | O_5^b(\lambda) | 0\rangle}{\Delta^2 - m_\pi^2} \right] U \\ & - \bar{U}' \left[\Gamma_5^{(V)}(p', p-k, \lambda)\, \tau^b + \frac{ß g_A}{2 f_\pi} \not{\Delta} \gamma_5 \tau^c \frac{\langle \pi^c(\Delta) | O_5^b(\lambda) | 0 \rangle}{\Delta^2 - m_\pi^2} \right] \frac{\not{p} + M}{2p\cdot k} \frac{ß g_A}{2 f_\pi} \not{k} \gamma_5 \tau^a U \\ & + \mathcal{O}(\varepsilon). \end{aligned} \tag{135}$$

To further simplify this expression, we need the pion-nucleon scattering amplitude which appears in the $\pi(\Delta)$ pole terms in the region of small momentum transfer. We can derive the soft-pion theorem for this amplitude from the identity

$$\frac{\Delta^\mu}{f_\pi} \langle N' \pi^a(k) | A_\mu^c(0) | N \rangle = -ß m_\pi^2 \langle N' \pi^a(k) | \Phi^c(0) | N \rangle. \tag{136}$$

For the matrix element $\langle N' \pi^a | A_\mu^c | N\rangle$ on the left hand side, we can use the soft-pion theorem (135) since

$$A_\mu^c(0) = 2 O_5^c(\lambda = 0). \tag{137}$$

The right hand side of (136) is dominated by the pion pole in Δ^2,

$$-ß m_\pi^2 \langle N' \pi^a(k) | \Phi^c(0) | N \rangle = \mathcal{M}(N'\pi^a(k)|N\pi^c(\Delta)) \frac{m_\pi^2}{\Delta^2 - m_\pi^2} + \mathcal{O}(\varepsilon^2). \tag{138}$$

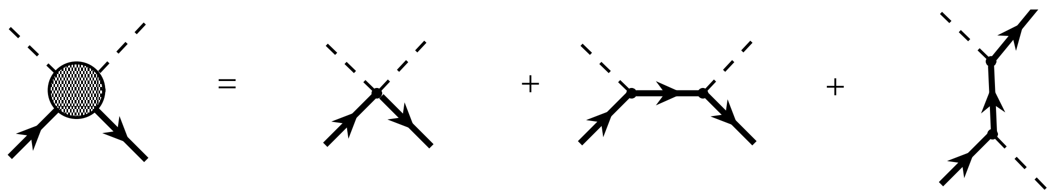

Figure 3: Leading contributions to the pion-nucleon scattering amplitude $\mathcal{M}(N(p')\,\pi(\Delta)|N(p)\,\pi(k))$ for soft pion momenta Δ and k. The pointlike vertex with four particles attached corresponds to the Weinberg coupling.

Thus, we can solve equation (136) for the pion scattering amplitude and obtain the well-known result

$$\begin{aligned}\mathcal{M}(N'\pi^a(k)|N\pi^c(\Delta)) &= -\frac{\varepsilon^{acd}}{4f_\pi^2}\bar{U}'(\not{k}+\not{\Delta})\tau^d U \\ &- \left(\frac{g_A}{f_\pi}\right)^2 \bar{U}'\left(\not{k}\gamma_5\frac{\tau^a}{2}\text{ß}\frac{\not{p}'+M}{2p'\cdot k}\not{\Delta}\gamma_5\frac{\tau^c}{2} - \not{\Delta}\gamma_5\frac{\tau^c}{2}\text{ß}\frac{\not{p}+M}{2p\cdot k}\not{k}\gamma_5\frac{\tau^a}{2}\right)U + \mathcal{O}(\varepsilon^2), \quad (139)\end{aligned}$$

see also Fig. 3. If we insert this expression into equation (135), we find that the double poles are canceled, and the final version of the soft-pion theorem reads

$$\begin{aligned}&\langle N'\pi^a(k)|O_5^b(\lambda)|N\rangle \\ &= \frac{\varepsilon^{abc}}{f_\pi}\bar{U}'\Gamma^{(V)}(p',p,\lambda)\,\tau^c U - \frac{\varepsilon^{acd}}{4f_\pi^2}\bar{U}'(\not{k}+\not{\Delta})\tau^d U\frac{\text{ß}}{\Delta^2-m_\pi^2}\langle\pi^c(\Delta)|O_5^b(\lambda)|0\rangle \\ &\quad+\frac{\text{ß}g_A}{2f_\pi}\bar{U}'\left[\not{k}\gamma_5\tau^a\frac{\not{p}'+M}{2p'\cdot k}\Gamma_5^{(V)}(p'+k,p,\lambda)\,\tau^b - \Gamma_5^{(V)}(p',p-k,\lambda)\,\tau^b\frac{\not{p}+M}{2p\cdot k}\not{k}\gamma_5\tau^a\right]U \\ &\quad+\mathcal{O}(\varepsilon), \qquad (140)\end{aligned}$$

see also figure 4.

Moderate momentum transfer In the region of moderate momentum transfer, i.e. $-\Delta^2 \gg \varepsilon^2$, the pion pole term in the soft-pion theorem (140) is negligible as it should since according to the Dirac equation for the nucleon spinors we have the suppression factor

$$\bar{U}'(\not{k}+\not{\Delta})U = 2\bar{U}'\not{k}U = \mathcal{O}(\varepsilon), \qquad (141)$$

while the denominator $\Delta^2 - m_\pi^2$ is no longer small. Therefore, in this region the soft-pion theorem again agrees with the corresponding one in Guichon et al. [8].

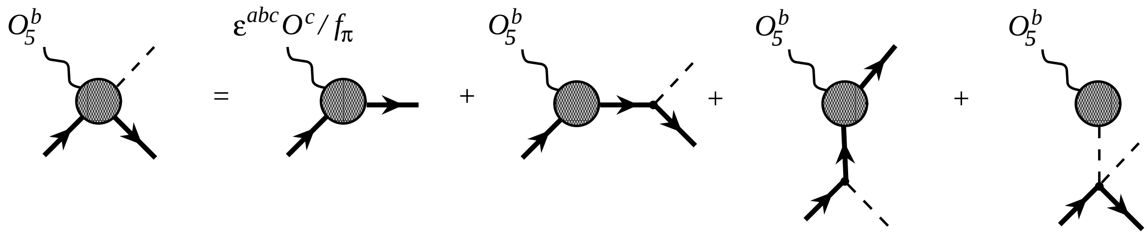

Figure 4: The soft-pion theorem for the matrix element $\langle N(p')\,\pi^a(k)|O_5^b(\lambda)|N(p)\rangle$, where $O_5^b(\lambda)$ is the operator $\bar{\psi}(-\lambda n/2)\,\not{n}\gamma_5\tau^b\psi(\lambda n/2)$.

4.2.4 Isoscalar operator of axial vector type

In the case of the isoscalar axial quark operator, matters are simpler because commutator and pion pole terms vanish and we are simply left with the nucleon pole terms:

$$\begin{aligned}
&\langle N'\pi^a(k)|\bar{\psi}(-\lambda n/2)\not{n}\gamma_5\psi(\lambda n/2)|N\rangle \\
&= \frac{\text{\ss} g_A}{2f_\pi}\bar{U}'\left[\not{k}\gamma_5\tau^a\frac{\not{p}'+M}{2p'\cdot k}\Gamma_5^{(S)}(p'+k,p,\lambda)-\Gamma_5^{(S)}(p',p-k,\lambda)\frac{\not{p}+M}{2p\cdot k}\not{k}\gamma_5\tau^a\right]U, \\
&\quad +\mathcal{O}(\varepsilon),
\end{aligned} \tag{142}$$

where

$$\Gamma_5^{(S)}(p_2,p_1,\lambda)=\int\limits_{-1}^{1}\mathrm{d}x\,\mathrm{e}^{-\text{\ss}\lambda x\,n\cdot(p_1+p_2)/2}\left[\tilde{H}^{(S)}\not{n}+\tilde{E}^{(S)}\frac{n\cdot(p'-p)}{2M}\right]\gamma_5. \tag{143}$$

4.3 Application to the pion-nucleon distributions

From the soft-pion theorems for the matrix elements of twist-2 lightcone operators, equations (110), (114), (115), (140), and (142), we arrive at the corresponding results for the πN distributions H_i and $\tilde{H}_i$ by Fourier transformation with respect to the light-cone coordinate λ and an appropriate decomposition of the Dirac matrix structure. These results are presented in the following at the pion threshold. The general (non-threshold) results are given in appendix B.

The soft-pion theorem (110) leads to

$$H_{\mathrm{1th}}^{(+)}(x,\xi,t) = -\frac{2M^2(1-\xi)}{2M^2-t}H^{(V)}(x,\xi,\Delta^2)+\frac{4M^2\xi-t}{2(2M^2-t)}E^{(V)}(x,\xi,\Delta^2) \tag{144}$$

$$H_{\mathrm{2th}}^{(+)}(x,\xi,t) = \frac{t[H^{(V)}(x,\xi,\Delta^2)+E^{(V)}(x,\xi,\Delta^2)]}{2(2M^2-t)} \tag{145}$$

and

$$
\begin{aligned}
H^{(-)}_{1\mathrm{th}}(x,\xi,t) &= \frac{\xi_0}{g_A}[\tilde{E}^{(V)}(x,\xi_0,t) - \tilde{E}^{(V)}_{\mathrm{pole}}(x,\xi_0,t)] \\
&\quad + \frac{2M^2(1-\xi)}{2M^2-t} H^{(V)}(x,\xi,\Delta^2) - \frac{4M^2\xi - t}{2(2M^2-t)} E^{(V)}(x,\xi,\Delta^2) \\
&\quad + \frac{2M^2}{m_\pi^2 - t} H^{(V)}_\pi\left(\frac{x}{\bar{p}_t\cdot n}, \frac{\xi}{\bar{p}_t\cdot n}, 0\right)\,\theta(\bar{p}_t\cdot n - |x|) \qquad (146) \\
H^{(-)}_{2\mathrm{th}}(x,\xi,t) &= -\frac{\tilde{H}(x,\xi_0,t)}{g_A} - \frac{t[H^{(V)}(x,\xi,\Delta^2) + E^{(V)}(x,\xi,\Delta^2)]}{2(2M^2-t)}. \qquad (147)
\end{aligned}
$$

Here, Δ^2 is considered as function of t according to the threshold requirement (29). The variable ξ_0 is defined as

$$
\xi_0 = \frac{(p-p')\cdot n}{(p+p')\cdot n}, \qquad (148)
$$

its threshold value is related to ξ via

$$
\xi_0 \stackrel{\mathrm{th}}{=} \frac{(2M+m_\pi)\xi + m_\pi}{2M + m_\pi(1+\xi)}. \qquad (149)
$$

The average momentum in the t channel is referred to as $\bar{p}_t$,

$$
\bar{p}_t = \frac{p - p' + k}{2}. \qquad (150)
$$

At threshold, $\bar{p}_t \cdot n$ can be expressed in terms of ξ:

$$
\bar{p}_t\cdot n \stackrel{\mathrm{th}}{=} \xi + \frac{m_\pi(1-\xi)}{M+m_\pi}. \qquad (151)
$$

Further, $\tilde{E}^{(V)}_{\mathrm{pole}}$ denotes the pion-pole contribution to $\tilde{E}^{(V)}$ and is given in terms of the pion distribution amplitude ϕ_π as [18, 19, 20, 21]

$$
\tilde{E}^{(V)}_{\mathrm{pole}}(x,\xi_0,t) = \frac{(2M)^2 g_A}{m_\pi^2 - t}\frac{1}{\xi_0}\phi_\pi\left(\frac{x}{\xi_0}\right)\,\theta(\xi_0 - |x|). \qquad (152)
$$

Next, we turn to the πN distributions for the soft-pion theorems (114) and (115). They read

$$
\begin{aligned}
H^{(0,G)}_{1\mathrm{th}}(x,\xi,t) &= -\frac{2M^2(1-\xi)}{2M^2-t}\, H^{(S,G)}(x,\xi,\Delta^2) + \frac{4M^2\xi - t}{2(2M^2-t)}\, E^{(S,G)}(x,\xi,\Delta^2) \\
&\quad + \frac{2M^2}{m_\pi^2 - t}\, H^{(S,G)}_\pi\left(\frac{x}{\bar{p}_t\cdot n}, \frac{\xi}{\bar{p}_t\cdot n}, 0\right)\theta(\bar{p}_t\cdot n - |x|) \qquad (153) \\
H^{(0,G)}_{2\mathrm{th}}(x,\xi,t) &= \frac{t[H^{(S,G)}(x,\xi,\Delta^2) + E^{(S,G)}(x,\xi,\Delta^2)]}{2(2M^2-t)}, \qquad (154)
\end{aligned}
$$

where the first superscript refers to the isoscalar quark distributions and the second one to the gluon distributions, respectively.

Finally, we give the results related to the soft-pion theorems (140) and (142), again using a combined notation for the isoscalar (0) and isovector even (+) GPDs:

$$\tilde{H}^{(0,+)}_{1\mathrm{th}}(x,\xi,t) = \frac{2M^2(1-\xi)}{2M^2-t}\tilde{H}^{(S,V)}(x,\xi,\Delta^2) - \frac{t\xi\tilde{E}^{(S,V)}(x,\xi,\Delta^2)}{2(2M^2-t)} \tag{155}$$

$$\tilde{H}^{(0,+)}_{2\mathrm{th}}(x,\xi,t) = -\frac{4M^2-t}{2(2M^2-t)}\,\tilde{H}^{(S,V)}(x,\xi,\Delta^2) \tag{156}$$

and

$$\begin{aligned}\tilde{H}^{(-)}_{1\mathrm{th}}(x,\xi,t) &= \frac{E^{(V)}(x,\xi_0,t)}{g_A} - \frac{2M^2(1-\xi)}{2M^2-t}\tilde{H}^{(V)}(x,\xi,\Delta^2)\\ &\quad+\frac{t\xi\tilde{E}^{(V)}(x,\xi,\Delta^2)}{2(2M^2-t)} - \frac{2Mm_\pi}{g_A t}\phi_\pi\left(\frac{x}{\xi}\right)\,\theta(\xi-|x|)\end{aligned} \tag{157}$$

$$\tilde{H}^{(-)}_{2\mathrm{th}}(x,\xi,t) = \frac{4M^2-t}{2(2M^2-t)}\tilde{H}^{(V)}(x,\xi,\Delta^2) - \frac{E^{(V)}(x,\xi_0,t)+H^{(V)}(x,\xi_0,t)}{g_A}. \tag{158}$$

4.4 Results for the moments of πN distributions

In section 3.3, we have described how to obtain pion emission form factors of certain local operators from the moments of the πN distributions according to the polynomiality property. Now that we have derived the soft-pion theorems for the πN distributions, we can easily read off these form factors after taking the corresponding moment. In the cases where the results are known, this procedure provides a check of the previous calculations.

4.4.1 First moment of $H_i^{(0,\pm)}$

The soft-pion theorems for pion emission induced by the local vector current, i.e. for the form factors A_i ($i=1,\ldots,8$) that we have defined in formula (38), are given in appendix C. For a discussion, we restrict ourselves in the following to the pion threshold where the number of independent form factors is reduced to two. The conventional quantities for such a description are the transversal and longitudinal s wave multipoles $E^{(0,\pm)}$ and $L^{(0,\pm)}$. In the center-of-mass frame where $\vec{p}\,'=\vec{k}=0$ and $\vec{p}=-\vec{\Delta}$, these threshold multipoles can be defined using the spatial components of the matrix elements

$$\langle N_f(p',S')\,\pi^a(k)|\bar{\psi}\gamma_\mu\left\{\begin{array}{c}1\\ \tau^b\end{array}\right\}\psi|N_i(p,S)\rangle = \text{ß}\left\{\begin{array}{c}T^{\mu(0)}_{S'S}\,\tau^a_{fi}\\ T^{\mu(+)}_{S'S}\delta^{ab}\delta_{fi}+T^{\mu(-)}_{S'S}\text{ß}\varepsilon^{abc}\tau^c_{fi}\end{array}\right\},$$

namely

$$\frac{e}{6}\vec{T}^{(0)}_{S'S} = 8\pi(M+m_\pi)\left[E^{(0)}_{0+}\vec{\sigma}_{S'S}+(L^{(0)}_{0+}-E^{(0)}_{0+})\frac{\vec{p}\,\vec{p}\cdot\vec{\sigma}_{S'S}}{|\vec{p}|^2}\right] \tag{159}$$

$$\frac{e}{2}\vec{T}^{(\pm)}_{S'S} = 8\pi(M+m_\pi)\left[E^{(\pm)}_{0+}\vec{\sigma}_{S'S}+(L^{(\pm)}_{0+}-E^{(\pm)}_{0+})\frac{\vec{p}\,\vec{p}\cdot\vec{\sigma}_{S'S}}{|\vec{p}|^2}\right], \tag{160}$$

where e is the electromagnetic coupling constant and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli spin matrices.

According to these definitions, we obtain for the relations between these multipoles and the πN distributions at threshold

$$E_{0+}^{(0)} = -\frac{ec}{3}\int_{-1}^{1} \mathrm{d}x\, H_{2\mathrm{th}}^{(0)}(x,\xi,t) = \frac{ec}{3}\left[-\frac{tG_M^{(S)}(\Delta^2)}{2(2M^2-t)} + \mathcal{O}(\varepsilon)\right] \tag{161}$$

$$E_{0+}^{(+)} = -ec\int_{-1}^{1} \mathrm{d}x\, H_{2\mathrm{th}}^{(+)}(x,\xi,t) = ec\left[-\frac{tG_M^{(V)}(\Delta^2)}{2(2M^2-t)} + \mathcal{O}(\varepsilon)\right] \tag{162}$$

$$E_{0+}^{(-)} = -ec\int_{-1}^{1} \mathrm{d}x\, H_{2\mathrm{th}}^{(-)}(x,\xi,t) = ec\left[F_A^{(V)}(t) + \frac{tG_M^{(V)}(\Delta^2)}{2(2M^2-t)} + \mathcal{O}(\varepsilon)\right], \tag{163}$$

and

$$L_{0+}^{(0)} = E_{0+}^{(0)} + \frac{ect}{12M^2}\int_{-1}^{1} \mathrm{d}x\, H_{1\mathrm{th}}^{(0)}(x,1,t) = \frac{ec}{3}\left[-\frac{tG_E^{(S)}(\Delta^2)}{2(2M^2-t)} + \mathcal{O}(\varepsilon)\right] \tag{164}$$

$$L_{0+}^{(+)} = E_{0+}^{(\pm)} + \frac{ect}{4M^2}\int_{-1}^{1} \mathrm{d}x\, H_{1\mathrm{th}}^{(+)}(x,1,t) = ec\left[-\frac{tG_E^{(V)}(\Delta^2)}{2(2M^2-t)} + \mathcal{O}(\varepsilon)\right] \tag{165}$$

$$L_{0+}^{(-)} = E_{0+}^{(-)} + \frac{ect}{4M^2}\int_{-1}^{1} \mathrm{d}x\, H_{1\mathrm{th}}^{(-)}(x,1,t) = ec\left[\frac{m_\pi^2 F_A^{(V)}(t)}{m_\pi^2 - t} + \frac{tG_E^{(V)}(\Delta^2)}{2(2M^2-t)} + \mathcal{O}(\varepsilon)\right], \tag{166}$$

with the kinematical prefactor

$$c = \frac{g_A}{16\pi f_\pi}\frac{\sqrt{4M^2-t}}{M+m_\pi}. \tag{167}$$

The functions

$$G_M(\Delta^2) = F_1(\Delta^2) + F_2(\Delta^2), \qquad G_E(\Delta^2) = F_1(\Delta^2) + \frac{\Delta^2}{4M^2}F_2(\Delta^2) \tag{168}$$

are the magnetic and electric form factors of the nucleon, with the superscripts S and V indicating the isovector and isoscalar combination, respectively. $F_A^{(V)} \equiv G_A^{(V)}/g_A$ denotes the isovector axial form factor normalized to unity. Figure 5 shows a plot of the multipoles E_{0+} and L_{0+} as function of the invariant momentum transfer Δ^2.

The multipoles $E_{0+}^{(0,+)}$ and $L_{0+}^{(0,+)}$ turn out to be quite small, so that the corrections which are neglected in the soft-pion theorems become important for these quantities. In the conventional units that are also used in figure 5, these corrections could be as large

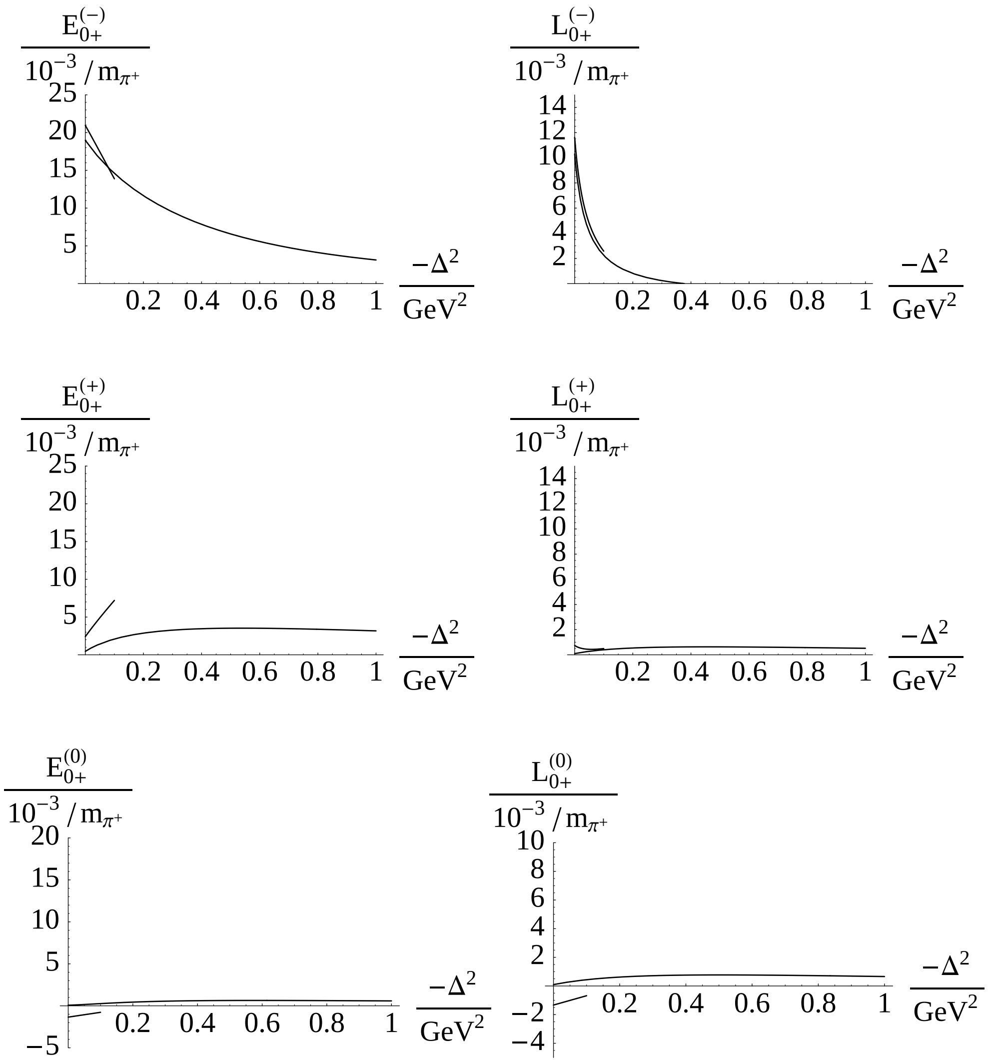

Figure 5: The transverse and longitudinal s-wave multipoles as function of $\Delta^2 = (1 + m_\pi/M)t + m_\pi^2$. The curves up to $-\Delta^2 = 1$ GeV^2 correspond to the soft-pion theorems (161) to (166). The curves up to $-\Delta^2 = 0.1$ GeV^2 show the one-loop ChPT results by Bernard et. al. [22] as given in their formula (5.1). The references for the nucleon form factors which have been used for this plot can be found in appendix A. For the pion electromagnetic form factor we have used the monopole form $F_\pi(\Delta^2) = (1 - \Delta^2\langle r_\pi^2\rangle/6)^{-1}$ with $\langle r_\pi^2\rangle \approx 0.45$ fm^2, see references [23, 24].

---

as $cm_\pi/M \sim 3 \times 10^{-3}/m_{\pi^+}$. The situation is different for the multipoles $E_{0+}^{(-)}$ and $L_{0+}^{(-)}$; they reach sizable values in particular at low momentum transfer, and corrections become relatively small.

Since the first use of soft-pion theorems in pion-electroproduction that can be dated back to the work of Nambu and Shrauner in 1962 [25], a lot of progress has been made in calculating corrections to the early soft-pion results. Within the framework of chiral perturbation theory, pion-electroproduction has been computed including the one-loop level [22]. The resulting threshold multipoles are also shown in figure 5. Unfortunately, these achievements of ChPT are limited to the region of very small momentum transfer, $-t \ll M^2$. If we expand the soft-pion results (161) to (166) for such small $-t$, we get correspondence with the leading-order results of ChPT.

4.4.2 First moment of $\tilde{H}_i^{(\pm)}$

Further, we would like to discuss the form factors of pion emission induced by the local isovector axial current. They can be obtained from the first moments of the πN distributions $\tilde{H}_i^{(\pm)}$. Again, we restrict ourselves to the threshold. Then the matrix element is parametrized in terms of three s-wave multipoles $L_{0+}^{(\pm)}$, $H_{0+}^{(\pm)}$, and $M_{0+}^{(\pm)}$:

$$\langle N_f(p', S')\, \pi^a(k) | A^{\mu a} | N_i(p, S)\rangle = \frac{\text{ß}}{2}(\delta^{ab}\delta_{fi}\tilde{T}^{\mu(+)}_{S'S} + \text{ß}\varepsilon^{abc}\tau^c_{fi}\tilde{T}^{\mu(-)}_{S'S}), \tag{169}$$

where in the center-of-mass frame ($\vec{p}\,' + \vec{k} = 0$) one defines for the time and space components of $\tilde{T}^\mu$

$$\tilde{T}^{0(\pm)}_{S'S} = 16\pi(M + m_\pi)(L^{(\pm)}_{0+} + \Delta_0 H^{(\pm)}_{0+})\delta_{S'S}, \tag{170}$$

$$\vec{\tilde{T}}^{(\pm)}_{S'S} = 16\pi(M + m_\pi)\left(-\vec{p} H^{(\pm)}_{0+}\delta_{S'S} + M^{(\pm)}_{0+}\text{ß}\frac{\vec{\sigma}_{S'S} \times \vec{p}}{|\vec{p}|}\right) \tag{171}$$

(see [26] and original references therein). The relations between these multipoles and the moments of theshold πN distributions are

$$L^{(\pm)}_{0+} = c\int\limits_{-1}^{1} \mathrm{d}x \left(\frac{M + m_\pi}{M}\tilde{H}^{(\pm)}_{1\mathrm{th}}(x, 0, t) + \frac{2M(2M + m_\pi)}{4M^2 - t}\tilde{H}^{(\pm)}_{2\mathrm{th}}(x, \xi, t)\right) \tag{172}$$

$$H^{(\pm)}_{0+} = \frac{c}{M}\int\limits_{-1}^{1} \mathrm{d}x \left(-\frac{1}{2}\tilde{H}^{(\pm)}_{1\mathrm{th}}(x, 1, t) - \frac{2M^2}{4M^2 - t}\tilde{H}^{(\pm)}_{2\mathrm{th}}(x, \xi, t)\right) \tag{173}$$

$$M^{(\pm)}_{0+} = -\frac{g_A}{16\pi f_\pi}\frac{\sqrt{-t}}{M + m_\pi}\int\limits_{-1}^{1} \mathrm{d}x\, \tilde{H}^{(\pm)}_2(x, \xi, t), \tag{174}$$

where c is the same kinematical factor as in definition (167). Note that in taking the moments of the πN distributions at a particular skewedness ξ, in general one has to integrate first and then to insert the value of ξ to avoid unphysical regions in the ξ-t

plane. After the insertion of the soft-pion theorems for the functions $\tilde{H}_{i\mathrm{th}}$, we obtain

$$
\begin{aligned}
M_{0+}^{(-)} &= \frac{g_A}{16\pi f_\pi}\sqrt{-\frac{t}{M^2}}\left[\frac{G_M^{(V)}(t)}{g_A} - \frac{(4M^2-t)G_A^{(V)}(t)}{2(2M^2-t)}\right] &(175)\\
M_{0+}^{(+)} &= \frac{g_A}{16\pi f_\pi}\sqrt{-\frac{t}{M^2}}\frac{(4M^2-t)G_A^{(V)}(t)}{2(2M^2-t)} &(176)
\end{aligned}
$$

and

$$
L_{0+}^{(-)} = -c\frac{4MG_E^{(V)}(t)}{4M^2-t}, \qquad H_{0+}^{(-)} = \frac{c}{M}\left[\frac{m_\pi/g_A}{\Delta^2-m_\pi^2} + \frac{2MG_E^{(V)}(t)}{(4M^2-t)g_A}\right], \tag{177}
$$

while $L_{0+}^{(+)}$ and $H_{0+}^{(+)}$ are vanishing within the accuracy of soft-pion theorems. At small momentum transfer, these results agree with the leading order expressions of the ChPT calculation in reference [26].

4.4.3 Second moment of $H_i^{(0)} + H_i^{(G)}/2$

As described in section 3.3, the second moment of the πN distributions $H_i^{(0)}$ and $H_i^{(G)}$ give the form factors B_i for soft pion emission induced by the energy-momentum tensor. The resulting soft-pion theorem is

$$
\begin{aligned}
\langle N'\pi^a|\mathcal{T}^{\mu\nu}|N\rangle &= \frac{ß g_A}{Mf_\pi}\sum_{i=1}^{20}\bar{U}'B_i\Gamma_i^{\mu\nu}\tau^a U\\
&= \frac{ß g_A}{Mf_\pi}\bar{U}'\left\{\left[\Delta^2\left(C-\frac{M^2}{m_\pi^2-t}\right)-M^2\right]\frac{g^{\mu\nu}}{4} + B\bar{p}^\mu\bar{p}^\nu - \frac{C}{4}\Delta^\mu\Delta^\nu + \frac{M^2k^\mu k^\nu}{4(m_\pi^2-t)}\right.\\
&\quad + \frac{2M^2(A+B)}{u-M^2}\bar{p}^{\{\mu}k^{\nu\}} - \frac{M^2\Delta^{\{\mu}k^{\nu\}}}{m_\pi^2-t} - \left(\frac{M}{W^2-M^2}+\frac{M}{u-M^2}\right)\not{k}\\
&\quad \left.\times\left[\frac{\Delta^2}{4}Cg^{\mu\nu} + B\bar{p}^\mu\bar{p}^\nu - \frac{C}{4}\Delta^\mu\Delta^\nu + (A+B)\bar{p}^{\{\mu}\gamma^{\nu\}}\right]\right\}\gamma_5\tau^a U + \mathcal{O}(\varepsilon), \qquad (178)
\end{aligned}
$$

where $A = A(\Delta^2)$ and so on for the other nucleon form factors B and C. For the determination of B_1 and B_8 from the πN distribution moments, it has been necessary to use the current conservation relations (47) and (50), because these form factors appear together with the metric tensor $g^{\mu\nu}$ which vanishes after contraction with the lightcone vectors $n^\mu n^\nu$. The remaining six current conservation constraints are fulfilled, so that

$$
\Delta_\mu\langle N'\pi^a|\mathcal{T}^{\mu\nu}|N\rangle = \mathcal{O}(\varepsilon). \tag{179}
$$

5 Hard pion production with additional soft pion

As an application of the presented soft pion theorems for the πN distributions, we consider the process of hard π^+ production off the proton with soft pion emission. The two possible reactions are

$$\gamma_L^*(q) + \mathrm{p}(p,S) \to \mathrm{n}(p',S') + \pi^0(k) + \pi^+(q'), \tag{180}$$

and

$$\gamma_L^*(q) + \mathrm{p}(p,S) \to \mathrm{p}(p',S') + \pi^-(k) + \pi^+(q'). \tag{181}$$

We shall calculate the longitudinal differential cross section as well as the transverse spin asymmetries for these processes, in comparison to the corresponding pure process, namely

$$\gamma_L^*(q) + \mathrm{p}(p,S) \to \mathrm{n}(p',S') + \pi^+(q'). \tag{182}$$

5.1 Longitudinal cross section

First, we recall the definition of the differential longitudinal cross section of π^+ production without soft pion emission, reaction (182):

$$\mathrm{d}^2\sigma_L^{(\mathrm{n})} = \frac{\overline{|\mathcal{M}_L^{(\mathrm{n})}|^2}\,\mathrm{d}^2\Phi(q+p;q',p')}{2(s-M^2)\sqrt{\Lambda(s,M^2,-Q^2)}} \tag{183}$$

where $\mathrm{d}^2\Phi$ is the two-particle phase space volume,

$$\mathrm{d}^2\Phi(q+p;q',p') = \frac{\mathrm{d}^3q'}{2q_0'}\frac{\mathrm{d}^3p'}{2p_0'}(2\pi)^4\delta(q+p-q'-p'), \tag{184}$$

and Λ denotes the conventional kinematical function

$$\Lambda(x,y,z) = \sqrt{(x-y-z)^2 - 4yz}. \tag{185}$$

Introducing the angle ϕ, which is the azimuthal angle of $\vec{q}\,'$ with respect to the direction of $\vec{q}$ in the center-of-mass frame, and $t_0 = (p-p')^2$, one obtains for the cross section

$$\frac{\mathrm{d}^2\sigma_L^{(\mathrm{n})}}{\mathrm{d}\phi\,\mathrm{d}t_0} = \frac{\overline{|\mathcal{M}_L^{(\mathrm{n})}|^2}}{32\pi^2(s-M^2)\Lambda(s,M^2,-Q^2)}. \tag{186}$$

The amplitude is given through the matrix element

$$\mathcal{M}_L^{(\mathrm{n})} = e\langle \mathrm{n}(p')\,\pi^+(q')|\varepsilon_L\cdot J|\mathrm{p}(p)\rangle \tag{187}$$

of the electromagnetic current

$$J = \bar{\psi}\gamma_\mu\left(\frac{1}{6}+\frac{\tau^3}{2}\right)\psi, \tag{188}$$

where ε_L is the longitudinal polarization vector, defined in the center-of-mass frame to be

$$\varepsilon_L = \frac{1}{Q}\left(|\vec{q}|, q_0\frac{\vec{q}}{|\vec{q}|}\right). \tag{189}$$

In the case of the reactions (180) and (181), i.e. with soft pion emission, we have to replace (184) with the three-particle phase space

$$\mathrm{d}^5\Phi(q+p;q',p',k) = \frac{\mathrm{d}^3q'}{2q'_0}\frac{\mathrm{d}^3p'}{2p'_0}\frac{\mathrm{d}^3k}{2k_0}(2\pi)^4\delta(q+p+k-q'-p'), \tag{190}$$

so that we obtain a differential cross section

$$\mathrm{d}^5\sigma_L^{(N\pi)} = \frac{\overline{|\mathcal{M}_L^{(N\pi)}|^2}\,\mathrm{d}^5\Phi(q+p;q',p',k)}{2(s-M^2)\sqrt{\Lambda(s,M^2,-Q^2)}} \tag{191}$$

defined analogously to equation (183). The amplitude is

$$\mathcal{M}_L^{(N\pi)} = e\langle N(p')\pi(k)\,\pi^+(q')|\varepsilon_L\cdot J|\mathrm{p}(p)\rangle, \tag{192}$$

and the superscript $N\pi$ labels the two final state possibilities $\mathrm{n}\pi^0$ or $\mathrm{p}\pi^-$. Integrating out the angular dependence of the soft pion and the invariant mass W^2 of the final nucleon-pion system up to some (not too large) value W^2_{max}, we further define

$$\frac{\mathrm{d}^2\sigma_L^{(N\pi)}}{\mathrm{d}\phi\,\mathrm{d}\Delta^2} \equiv \int\limits_{W^2_{\mathrm{th}}}^{W^2_{\mathrm{max}}}\mathrm{d}W^2\int\mathrm{d}\Omega_\pi\frac{\mathrm{d}^5\sigma_L^{(N\pi)}}{\mathrm{d}\phi\,\mathrm{d}\Delta^2\,\mathrm{d}W^2\mathrm{d}\Omega_\pi}. \tag{193}$$

Here, the variable Ω_π denotes the solid angle of the soft pion in the center-of-mass frame of the final nucleon-pion system. If the soft pion momentum k is sufficiently small, we can approximate the amplitude $\mathcal{M}_L^{(N\pi)}$ by its threshold value, and perform the phase-space integration in (193) to obtain

$$\frac{\mathrm{d}^2\sigma_L^{(N\pi)}}{\mathrm{d}\phi\,\mathrm{d}\Delta^2} = \frac{\overline{|\mathcal{M}_L^{(N\pi)}|^2}\,\Phi(W_{\mathrm{max}})}{32\pi^2(s-M^2)\Lambda(s,M^2,-Q^2)}, \tag{194}$$

with the phase-space function

$$\Phi(W_{\mathrm{max}}) = \frac{1}{6\pi^2}\sqrt{\frac{2Mm_\pi}{M+m_\pi}}(W_{\mathrm{max}}-W_{\mathrm{th}})^{3/2}\left[1+\mathcal{O}\left(\frac{W_{\mathrm{max}}}{W_{\mathrm{th}}}-1\right)\right]. \tag{195}$$

5.2 Transverse spin asymmetry

Besides the longitudinal cross section itself, there exist predictions for the so-called transverse spin asymmetry in π^+ production (182) with a polarized target proton [27, 28]. There were arguments brought forward that this observable is particularly useful, because it is less sensitive to higher-twist corrections and next-to-leading order corrections in the strong coupling α_s [29]. The definition of this asymmetry is

$$\mathcal{A}_{\mathrm{n}} = \frac{1}{|\vec{S}_\perp|}\left(\int\limits_0^\pi\mathrm{d}\phi\frac{\mathrm{d}^2\sigma_L^{(\mathrm{n})}}{\mathrm{d}\phi\,\mathrm{d}t_0} - \int\limits_\pi^{2\pi}\mathrm{d}\phi\frac{\mathrm{d}^2\sigma_L^{(\mathrm{n})}}{\mathrm{d}\phi\,\mathrm{d}t_0}\right)\left(\int\limits_0^{2\pi}\mathrm{d}\phi\frac{\mathrm{d}^2\sigma_L^{(\mathrm{n})}}{\mathrm{d}\phi\,\mathrm{d}t_0}\right)^{-1}, \tag{196}$$

where $\vec{S}_\perp$ is the component of the proton's spin vector that is transverse to $\vec{q}$ in the center-of-mass frame. A splitting of the squared amplitude into spin-dependent and spin-independent parts yields

$$\sum_{S'} |\mathcal{M}_L^{(\mathrm{n})}|^2 \propto s_0^{(\mathrm{n})}(x_B, t_0) + s_1^{(\mathrm{n})}(x_B, t_0)|\vec{S}_\perp| \sin\phi, \tag{197}$$

so that one finds

$$\mathcal{A}_\mathrm{n} = \frac{2 s_1^{(\mathrm{n})}}{\pi s_0^{(\mathrm{n})}}. \tag{198}$$

It is now straightforward to define an appropriate generalization of this observable for the equivalent process with soft pion emission:

$$\mathcal{A}_{N\pi} = \frac{1}{|\vec{S}_\perp|}\left(\int\limits_0^\pi \mathrm{d}\phi \frac{\mathrm{d}^2\sigma_L^{(N\pi)}}{\mathrm{d}\phi\,\mathrm{d}\Delta^2} - \int\limits_\pi^{2\pi} \mathrm{d}\phi \frac{\mathrm{d}^2\sigma_L^{(N\pi)}}{\mathrm{d}\phi\,\mathrm{d}\Delta^2}\right)\left(\int\limits_0^{2\pi} \mathrm{d}\phi \frac{\mathrm{d}^2\sigma_L^{(N\pi)}}{\mathrm{d}\phi\,\mathrm{d}\Delta^2}\right)^{-1} = \frac{2 s_1^{(N\pi)}}{\pi s_0^{(N\pi)}}, \tag{199}$$

where the functions $s_0^{(N\pi)}$ and $s_1^{(N\pi)}$ arise from the following decomposition of the threshold amplitude:

$$\sum_{S'} |\mathcal{M}_L^{(N\pi)}|^2 \propto \frac{1}{f_\pi^2}[s_0^{(N\pi)}(x_B, \Delta^2) + s_1^{(N\pi)}(x_B, \Delta^2)|\vec{S}_\perp| \sin\phi]. \tag{200}$$

The constant of proportionality not written explicitely equals that one in (197), it contains the distribution amplitude of the π^+ and the strong coupling α_s, for example. Supplementing the factor f_π^2 in the denominator serves to keep the functions $s_0^{(N\pi)}$ and $s_1^{(N\pi)}$ dimensionless.

Of course, from an experimental point of view, these asymmetries are useful only when the detection of the soft pion is guaranteed. If this is not the case, one should formulate an asymmetry with the cross sections of all three processes, (180) to (182), added up. In terms of the functions s_0 and s_1, such an asymmetry reads

$$\mathcal{A}_{\mathrm{n}+\mathrm{n}\pi^0+\mathrm{p}\pi^-} = \frac{2}{\pi}\frac{s_1^{(\mathrm{n})} + (s_1^{(\mathrm{n}\pi^0)} + s_1^{(\mathrm{p}\pi^-)})\Phi(W_\mathrm{max})/f_\pi^2}{s_0^{(\mathrm{n})} + (s_0^{(\mathrm{n}\pi^0)} + s_0^{(\mathrm{p}\pi^-)})\Phi(W_\mathrm{max})/f_\pi^2}. \tag{201}$$

5.3 Amplitude at threshold

Now we come to the amplitude $\mathcal{M}_L^{(N\pi)}$ of hard pion production with soft pion emission. For the sake of generality, we give the amplitude for arbitrary pion isospins a and c:

$$\begin{aligned}
& e\langle N(p')\pi^a(k)\,\pi^c(q')|\varepsilon_L\cdot J|N(p)\rangle \\
= \quad & -\text{ß} e \frac{2}{9}\frac{4\pi\alpha_s}{Q}\int\limits_{-1}^{1}\mathrm{d}u \frac{f_\pi\phi_\pi(u)}{1+u}\int\limits_{-1}^{1}\mathrm{d}x \frac{\text{ß} g_A}{M f_\pi}\bar{U}'[c^+\delta^{3c}\tau^a(\tilde{H}_{1\mathrm{th}}^{(0)} + \tilde{H}_{2\mathrm{th}}^{(0)} M \not{n}) \\
& +(c^+\delta^{ac}/3 + c^-\text{ß}\varepsilon^{c3a})(\tilde{H}_{1\mathrm{th}}^{(+)} + \tilde{H}_{2\mathrm{th}}^{(+)} M\hat{n}) \\
& +(c^+\text{ß}\varepsilon^{acd}\tau^d/3 + c^-(\delta^{ac}\tau^3 - \delta^{a3}\tau^c)(\tilde{H}_{1\mathrm{th}}^{(-)} + \tilde{H}_{2\mathrm{th}}^{(-)} M\hat{n})]\gamma_5 U,
\end{aligned} \tag{202}$$

where we abbreviate

$$c^{\pm} = \frac{1}{x - \xi + \text{ß}0} \pm \frac{1}{x + \xi - \text{ß}0}. \tag{203}$$

Evaluating this expression for the isospin combinations needed in π^+ production, and writing the amplitudes with the help of two functions $A_{N\pi}$ and $C_{N\pi}$ in the form

$$\mathcal{M}_L^{(N\pi)} = \frac{4\sqrt{2}e}{9} \frac{4\pi\alpha_s}{Q} \int_{-1}^{1} \mathrm{d}u \frac{\phi_\pi(u)}{1+u}\, \bar{u}(p') \left(A_{N\pi}\hat{n} + \frac{C_{N\pi}}{M} \right) \gamma_5 u(p), \tag{204}$$

we obtain

$$A_{\mathrm{n}\pi^0} = -g_A \int_{-1}^{1} \mathrm{d}x \left(\frac{2/3}{x - \xi + \text{ß}0} - \frac{1/3}{x + \xi - \text{ß}0} \right) \tilde{H}_{2\mathrm{th}}^{(-)}(x, \xi, t) \tag{205}$$

$$C_{\mathrm{n}\pi^0} = -g_A \int_{-1}^{1} \mathrm{d}x \left(\frac{2/3}{x - \xi + \text{ß}0} - \frac{1/3}{x + \xi - \text{ß}0} \right) \tilde{H}_{1\mathrm{th}}^{(-)}(x, \xi, t) \tag{206}$$

for the $\mathrm{n}\pi^0$ final state, and

$$A_{\mathrm{p}\pi^-} = \frac{g_A}{\sqrt{2}} \int_{-1}^{1} \mathrm{d}x \left(\frac{2/3}{x - \xi + \text{ß}0} - \frac{1/3}{x + \xi - \text{ß}0} \right) [\tilde{H}_{2\mathrm{th}}^{(+)}(x, \xi, t) + \tilde{H}_{2\mathrm{th}}^{(-)}(x, \xi, t)] \tag{207}$$

$$C_{\mathrm{p}\pi^-} = \frac{g_A}{\sqrt{2}} \int_{-1}^{1} \mathrm{d}x \left(\frac{2/3}{x - \xi + \text{ß}0} - \frac{1/3}{x + \xi - \text{ß}0} \right) [\tilde{H}_{1\mathrm{th}}^{(+)}(x, \xi, t) + \tilde{H}_{1\mathrm{th}}^{(-)}(x, \xi, t)] \tag{208}$$

for the $\mathrm{p}\pi^-$ final state.

Squaring the amplitude and summing over the final nucleon spin S', we obtain the functions $s_0^{(N\pi)}$ and $s_1^{(N\pi)}$ that we have introduced within the decomposition of the squared amplitude (200):

$$\begin{aligned} s_0^{(N\pi)} &= 4[(\mathrm{Re}A_{N\pi})^2 + (\mathrm{Im}A_{N\pi})^2] \frac{M(1-\xi^2)}{M+m_\pi} \\ &\quad +4\frac{2\xi + m_\pi(1+\xi)}{M+m_\pi} \mathrm{Re}(A_{N\pi}C_{N\pi}^*) - \frac{t}{M^2}[(\mathrm{Re}C_{N\pi})^2 + (\mathrm{Im}C_{N\pi})^2] \end{aligned} \tag{209}$$

$$s_1^{(N\pi)} = -4\frac{\sqrt{-\Delta_\perp^2}}{M+m_\pi} \mathrm{Im}(A_{N\pi}C_{N\pi}^*). \tag{210}$$

Note that when setting the pion mass equal to zero everywhere in the last expression, we recover the known formula for the reaction without soft pion emission.

5.4 Numerical results

5.4.1 Ratio of the longitudinal cross sections

The longitudinal cross section for usual hard π^+ production to leading twist and leading order in QCD was given in references [31, 27]. Here, we present the ratios of the unpolarized cross sections

$$\frac{\mathrm{d}\sigma_L^{(N\pi)}}{\mathrm{d}\Delta^2} \equiv \int_0^{2\pi} \mathrm{d}\phi \frac{\mathrm{d}^2\sigma_L^{(N\pi)}}{\mathrm{d}\phi\, \mathrm{d}\Delta^2} \tag{211}$$

and their corresponding counterpart without soft pion emission, respectively. These ratios must be calculated for a fixed direction of $\vec{q}\,'$, which implies the following relation between two- and three-particle final state variables Δ^2 and t_0:

$$\Delta^2 = -\frac{x_B(W_{\mathrm{th}}^2 - M^2 + t_0) - t_0}{1 - x_B} + \mathcal{O}\left(\frac{1}{Q^2}\right). \tag{212}$$

With this specification of Δ^2, we can define the ratios as functions of x_B and t_0:

$$R_{N\pi}(x_B, t_0, W_{\max}) = \frac{\mathrm{d}\sigma_L^{(N\pi)}}{\mathrm{d}\Delta^2}\left(\frac{\mathrm{d}\sigma_L^{(\mathrm{n})}}{\mathrm{d}t_0}\right)^{-1} = \frac{s_0^{(N\pi)}(x_B, \Delta^2)}{s_0^{(\mathrm{n})}(x_B, t_0)}\frac{\Phi(W_{\max})}{f_\pi^2}. \tag{213}$$

Concerning the models for the GPDs that enter the calculations of $s_0^{(N\pi)}$ and $s_0^{(\mathrm{n})}$, we refer to the review [1]; relevant formulas are summarized in appendix A. For the pion distribution amplitude, throughout we use the asymptotic form

$$\phi_\pi(u) = \phi_{\mathrm{as}}(u) = \frac{3}{4}(1 - u^2). \tag{214}$$

Figure 6 shows the ratio $R_{N\pi}$ as a function of x_B for three different values of t_0. We find that for a large range of t_0, the soft-pion contamination is roughly 10% for each individual channel. In situations where we have to add up both contributions, we arrive at a soft pion contamination that accounts for 15% to 30% within the presented range of t_0. The $W_{\max}$-dependence near threshold can easily be deduced from our results given at $W_{\max} = 1.15\ \mathrm{GeV}^2$, because it is incorporated explicitely in (213) through the phase space function $\Phi(W_{\max})$ (195).

5.4.2 Transverse spin asymmetry

The results for the transverse spin asymmetries of the processes with soft pion emission (180) and (181) are shown in figure 7. The values for these asymmetries turn out to be very small compared to those of the pure process (182), see figure 8. The small size of the asymmetries $\mathcal{A}_{N\pi}$ can be traced back to a particularly small value of $s_1^{(N\pi)}$ that arises from a significant cancellation within the following terms:

$$s_1^{(N\pi)} \propto \mathrm{Im}A_{N\pi}\mathrm{Re}C_{N\pi} - \mathrm{Re}A_{N\pi}\mathrm{Im}C_{N\pi}. \tag{215}$$

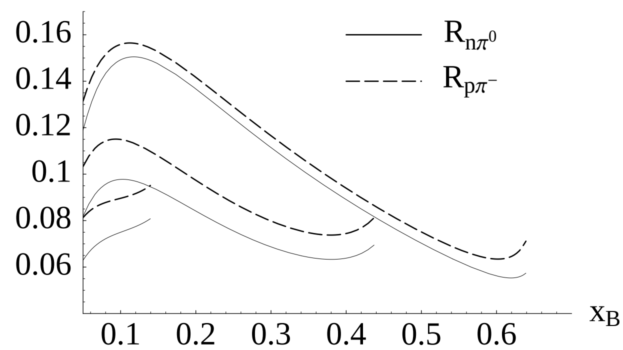

Figure 6: The two ratios of the unpolarized longitudinal cross sections with and without soft pion emission, $R_{\mathrm{n}\pi^0}$ and $R_{\mathrm{p}\pi^-}$, as functions of x_B for the values $t_0 = -1\ \mathrm{GeV}^2$, $t_0 = -0.3\ \mathrm{GeV}^2$ and $t_0 = -0.02\ \mathrm{GeV}^2$. Lower curves belong to smaller values of $-t_0$. The photon virtuality is $Q^2 = 10\ \mathrm{GeV}^2$ and $W_{\max} = 1.15$ GeV. The curves are plotted up to the maximally allowed value of the Bjorken variable, which is $x_{B\max} = 2/(1 + \sqrt{1 - 4M^2/t_0})$.

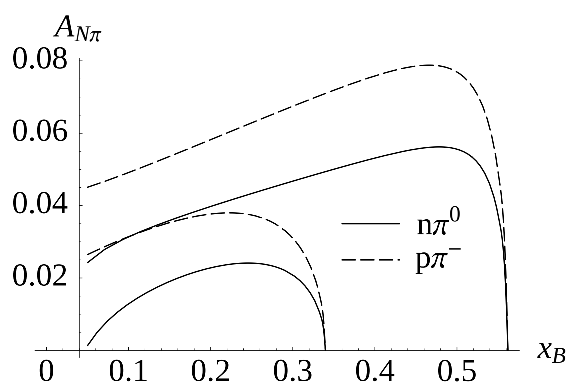

Figure 7: The transverse spin asymmetries of the processes $p + \gamma^* \to n + \pi^0_{\mathrm{soft}} + \pi^+$ and $p + \gamma^* \to p + \pi^-_{\mathrm{soft}} + \pi^+$ for the momentum transfers $t_0 = -0.3\ \mathrm{GeV}^2$ (lower curves) and $t_0 = -1\ \mathrm{GeV}^2$ (upper curves). The photon virtuality is chosen as $Q^2 = 10\ \mathrm{GeV}^2$.

In contrast, such a cancellation does not happen in the calculation of $s_1^{(\mathrm{n})}$, because in the pion-pole model for $\tilde{E}$, (see appendix A) the imaginary part of C_{n} vanishes, whereas A_{n}

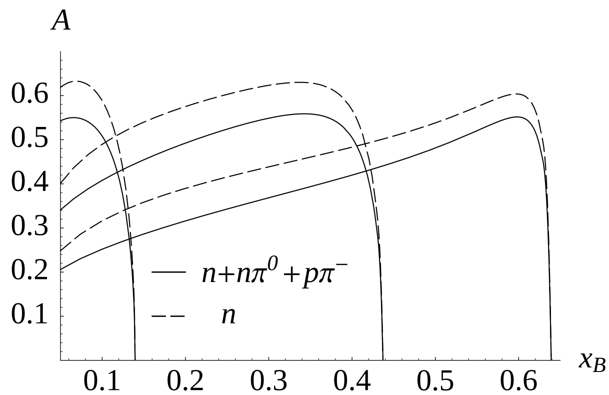

Figure 8: The transverse spin asymmetry for the pure hard π^+ production, $\mathcal{A}_{\mathrm{n}}$ (dashed curve), and for the hard π^+ production with soft pion admixture, $\mathcal{A}_{\mathrm{n+n}\pi^0+\mathrm{p}\pi^-}$ (solid curve). The values of the momentum transfer are $t_0 = -0.02$ GeV^2 (left), $t_0 = -0.3$ GeV^2 (middle) and $t_0 = -1$ GeV^2 (right). In the case of $\mathcal{A}_{\mathrm{n+n}\pi^0+\mathrm{p}\pi^-}$, the maximal invariant mass is chosen to be $W_{\mathrm{max}} = 1.15$ GeV. The photon virtuality is $Q^2 = 10$ GeV^2.

---

has non-negligible real and imaginary parts,

$$A_{\mathrm{n}} \;=\; -\int\limits_{-1}^{1} \mathrm{d}x \left(\frac{2/3}{x-\xi+\text{ß}0} - \frac{1/3}{x+\xi-\text{ß}0} \right) \tilde{H}^{(V)}(x,\xi,\Delta^2), \tag{216}$$

$$C_{\mathrm{n}} \;=\; \xi \int\limits_{-1}^{1} \mathrm{d}x \left(\frac{2/3}{x-\xi+\text{ß}0} - \frac{1/3}{x+\xi-\text{ß}0} \right) \tilde{E}^{(V)}(x,\xi,\Delta^2) = -\frac{3}{2} G_P^{(V)}(\Delta^2). \tag{217}$$

Moreover, the values of $s_0^{(N\pi)}$ are larger than those of $s_0^{(\mathrm{n})}$, roughly speaking by a factor two or three. Since they occur in the denominator, $\mathcal{A}_{N\pi} = 2s_1^{(N\pi)}/(\pi s_0^{(N\pi)})$, this gives a further reduction of the asymmetry $\mathcal{A}_{N\pi}$.

Let us now come to the asymmetry $\mathcal{A}_{\mathrm{n+n}\pi^0+\mathrm{p}\pi^-}$ of the combined processes, see equation (201). Our discussion of the functions s_0 and s_1 above already indicates that the inclusion of soft pions leads to an certain reduction of the asymmetry. This result is shown in figure 8. For an invariant mass integrated up to $W_{\mathrm{max}} = 1.15$ GeV, we find a downwards shift of the curves of about 10%, respectively.

6 Summary

In the following, we summarize our statements about soft pion emission in hard exclusive reactions:

- We have formulated a *parametrization* of twist-2 lightcone operators between initial nucleon and final nucleon-pion state. It has turned out that four functions, which we call *πN parton distributions*, are needed for each of these matrix elements. Further, we have shown how this number is reduced to two when the pion is exactly at the production threshold.

- Next, we demonstrated on several examples that the *moments of the πN distributions are polynomials* in the skewedness variables ξ and α. The coefficients in these polynomials are essentially the form factors (or, following the terminology of electroweak pion production, the "invariant amplitudes") of pion emission induced by corresponding *local* twist-2 operators.

- We have derived *soft-pion theorems for the πN distributions* which represent the leading terms in an expansion in small pion momentum and mass at given momentum transfer. The nontrivial ingredients of the final expressions are nucleon and pion GPDs as well as the pion distribution amplitude. If the momentum transfer is fixed to be large compared to the pion mass, we find that our results agree with Guichon et. al. [8]. In the opposite case, i.e. when the momentum transfer is small, we have argued on the example of certain moments that our results are consistent with the leading order of ChPT.

- Moreover, we have provided explicitely a parametrization and the soft-pion theorems for the resulting form factors of *pion emission from the nucleon induced by the energy-momentum tensor.*

- Finally, we have given analytical results for the *amplitude of hard pion production with soft pion emission* and some numerical estimates for the particular case of hard π^+ production using certain GPD models. In the presented kinematical region ($x_B \geq 0.05$, $0.02 \leq -\Delta_0^2/\mathrm{GeV}^2 \leq 1$, $W_{\max} = 1.15$ GeV, $Q^2 = 10$ GeV^2), we have obtained that the *contamination of the longitudinal cross section* of the reaction $\gamma^* + \mathrm{p} \rightarrow \pi^+ + \mathrm{n}$ caused by soft pions amounts to roughly 10% within each of the two possible channels $\gamma^* + \mathrm{p} \rightarrow \pi^+ + \mathrm{n} + \pi^0_{\mathrm{soft}}$ and $\gamma^* + \mathrm{p} \rightarrow \pi^+ + \mathrm{p} + \pi^-_{\mathrm{soft}}$. The effect of the soft-pion channels on the *transverse spin asymmetry* is a downwards shift of roughly 10%. Further, we have observed that the transverse spin asymmetry of the individual soft-pion reactions is about one order of magnitude smaller than that one which is predicted for the familiar process without soft pion.

We conclude that soft-pion theorems can provide useful estimates of the soft-pion contamination in hard exclusive reactions. The numerical studies for the contamination of hard π^+ production indicate that for an accurate interpretation of experiments an appropriate separation of the soft-pion channels is required. Second, we point out that in principle, according to the soft-pion theorems, hard exclusive reactions with soft pion emission can serve as an additional source to extract information about GPDs. Therefore, it is worth to consider corresponding experiments as well as further theoretical studies.

Acknowledgments

We acknowledge useful discussions with N. Kivel, and M. Vanderhaeghen. This work is supported in parts by the Sofja Kovalevskaja Programme of the Alexander von Humboldt Foundation, by BMBF and DFG.

Appendix

A GPD models

In this appendix, we summarize the models that we have used for the isovector nucleon GPDs which enter the presented calculations of hard π^+ production with soft pion emission. They are taken from reference [1], to which we refer for further explanations and motivations.

A.1 Isovector GPD $H^{(V)}$

The isovector and isoscalar GPDs $H^{(V)}$ and $H^{(S)}$ are linear combinations of the up and down quark GPDs in the proton,

$$H^{(V)} = H^u - H^d, \qquad H^{(S)} = H^u + H^d. \tag{218}$$

The models for the quark GPDs H^u and H^d contain the so-called double distribution term and the D-term. The latter is supposed to be flavor-independent, hence it cancels in the isovector combination. Further, one assumes a factorized ansatz for the quark GPDs H^q, $q = u, d$, so that

$$H^{(V)}(x,\xi,t) = H^u_{DD}(x,\xi)\frac{F_1^{u/p}(t)}{2} - H^d_{DD}(x,\xi)F_1^{d/p}(t), \tag{219}$$

where the double distribution representation is

$$H^q_{DD}(x,\xi) = \int\limits_{-1}^{1} \mathrm{d}\beta \int\limits_{-(1-|\beta|)}^{1-|\beta|} \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi)\, h(\beta,\alpha)\, H^q(\beta,0,0). \tag{220}$$

Here, the profile function h is chosen as

$$h(\beta,\alpha) = \frac{3}{4}\frac{(1-|\beta|)^2-\alpha^2}{(1-|\beta|)^3}, \tag{221}$$

and $H^q(x,0,0)$ denotes the forward limit that is related to the usual quark distribution functions $q(x)$ and antiquark distribution functions $\bar{q}(x)$ according to

$$H^q(x,0,0) = q(x)\,\theta(x) - \bar{q}(-x)\,\theta(-x). \tag{222}$$

For the numerical calculations, we use the leading order parametrization MRST2001LO from reference [32]. The second ingredient of the factorized ansatz for H is the Dirac form factor $F_1^{q/f}$. It is defined according to

$$\langle N_f(p')|\bar{\psi}_q\gamma_\mu\psi_q|N_f(p)\rangle = \bar{U}'\left[F_1^{q/f}\gamma_\mu + F_2^{q/f}\frac{\text{ß}\sigma_{\mu\nu}(p'-p)^\nu}{2M}\right]U, \qquad f = p,n, \tag{223}$$

and related to the nucleon Sachs form factors via

$$
\begin{aligned}
G_M^{(f)}(t) &= \frac{2}{3}[F_1^{u/f}(t) + F_2^{u/f}(t)] - \frac{1}{3}[F_1^{d/f}(t) + F_2^{d/f}(t)] \qquad (224)\\
G_E^{(f)}(t) &= \frac{2}{3}\left[F_1^{u/f}(t) + \frac{tF_2^{u/f}(t)}{4M^2}\right] - \frac{1}{3}\left[F_1^{d/f}(t) + \frac{tF_2^{d/f}(t)}{4M^2}\right]. \qquad (225)
\end{aligned}
$$

As input for numerical calculations, we have used the empirical fits of Brash et. al. [33] for the proton Sachs form factors and those of Bosted [34] for the neutron ones.

A.2 Isovector GPD $E^{(V)}$

For $E^{(V)}$, we take the very simple model

$$
E^{(V)}(x,\xi,t) = [E^u_{DD}(x,\xi) - E^d_{DD}(x,\xi)]G_D(t) \qquad (226)
$$

with the double distribution representation

$$
E^q_{DD}(x,\xi) = \int_{-1}^{1} \mathrm{d}\beta \int_{-(1-|\beta|)}^{1-|\beta|} \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi)\, E^q(\beta,0,0)\, h(\beta,\alpha). \qquad (227)
$$

Concerning the forward limit of E^q, the ansatz

$$
E^u(x,0,0) = \frac{\kappa^u}{2} u_v(x)\,\theta(x), \qquad E^d(x,0,0) = \kappa^d d_v(x)\,\theta(x) \qquad (228)
$$

with

$$
\kappa^u = 2\kappa^p + \kappa^n \qquad \kappa^d = \kappa^p + 2\kappa^n
$$

is assumed. Here, $\kappa^p = 1.793$ and $\kappa^n = -1.913$ are the anomalous magentic moments of the proton and neutron, and $q_v(x)$ denotes the valence quark distribution $q(x) - \bar{q}(x)$. In the t-dependent factor of the ansatz for $E^{(V)}$, G_D is the dipole form factor

$$
G_D(t) = \frac{1}{(1 - t/0.71\,\mathrm{GeV}^2)^2}. \qquad (229)
$$

A.3 Isovector GPD $\tilde{H}^{(V)}$

The GPD $\tilde{H}^{(V)} = \tilde{H}^u - \tilde{H}^d$ is modeled by the double distribution ansatz

$$
\tilde{H}^q(x,\xi,t) = \tilde{H}^q_{DD}(x,\xi)\, F_A^q(t) \qquad (230)
$$

with

$$
\tilde{H}^q_{DD}(x,\xi) = \int_{-1}^{1} \mathrm{d}\beta \int_{-(1-|\beta|)}^{1-|\beta|} \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi)\, h(\beta,\alpha)\, \Delta q_v(\beta), \qquad (231)
$$

where $\Delta q_v = [\Delta q(x) - \Delta\bar{q}(x)]\theta(x)$ is the polarized valence quark distribution. For the numerical input of Δq_v we have used the leading order analysis LSS2001LO given in reference [35]. Finally, the axial form factors are approximated by the dipole form of $F_A^{(V)}(t)$,

$$F_A^{(V)}(t) = \frac{G_A^{(V)}(t)}{g_A} = \frac{1}{(1 - t/M_A^2)^2} \tag{232}$$

with an axial mass $M_A \approx 1$ GeV.

A.4 Isovector GPD $\tilde{E}^{(V)}$

The GPD $\tilde{E}^{(V)}$ is modeled by the pion pole form

$$\tilde{E}^{(V)} = G_P^{(V)}(t)\frac{1}{\xi}\phi_\pi(x/\xi)\,\theta(\xi - |x|), \tag{233}$$

where the pseudoscalar form factor is assumed to be

$$G_P^{(V)}(t) = \frac{(2M)^2 G_A^{(V)}(t)}{m_\pi^2 - t} \tag{234}$$

and for the pion distribution amplitude ϕ_π, we use its asymptotic form $\phi_{\rm as}$ already given in (214).

B Soft-pion theorems for non-threshold pion-nucleon distributions

In section 4.3, we have shown the soft-pion theorems for the threshold πN distributions. Here, we give the corresponding general results, i.e. when the pion slightly deviates from the threshold. Throughout the following expressions, we shall imply the relations

$$u = 2M^2 + m_\pi^2 - W^2 - t + \Delta^2, \tag{235}$$

$$\bar{\alpha} = \alpha(1 - \xi), \tag{236}$$

$$\xi_0 = \frac{(p - p')\cdot n}{(p + p')\cdot n} = \frac{2\xi + \bar{\alpha}}{2 - \bar{\alpha}} = \frac{2\xi + \bar{\alpha}}{2}[1 + \mathcal{O}(\varepsilon)], \tag{237}$$

and

$$\bar{p}_t \cdot n = \frac{(p - p' + k)\cdot n}{2} = \xi + \bar{\alpha}, \tag{238}$$

which make the dependence of the πN distributions $H_i^{(0,\pm)}$ on the set of variables x, ξ, Δ^2, α, t, and W^2 explicit.

B.1 Isovector vector πN distributions

$$H_1^{(+)} = E^{(V)}(x,\xi,\Delta^2) + \frac{2M^2\bar{\alpha}}{u-M^2}[H^{(V)}(x,\xi,\Delta^2) + E^{(V)}(x,\xi,\Delta^2)] \tag{239}$$

$$H_2^{(+)} = 0 \tag{240}$$

$$H_3^{(+)} = -\left(\frac{M^2}{W^2-M^2} + \frac{M^2}{u-M^2}\right) E^{(V)}(x,\xi,\Delta^2) \tag{241}$$

$$H_4^{(+)} = -\left(\frac{M^2}{W^2-M^2} + \frac{M^2}{u-M^2}\right) [H^{(V)}(x,\xi,\Delta^2) + E^{(V)}(x,\xi,\Delta^2)], \tag{242}$$

$$\begin{aligned} H_1^{(-)} &= \frac{\xi_0}{g_A}[\tilde{E}^{(V)}(x,\xi_0,t) - \tilde{E}^{(V)}_{\mathrm{pole}}(x,\xi_0,t)] - \frac{2M^2\bar{\alpha}}{u-M^2}[H^{(V)}(x,\xi,\Delta^2) + E^{(V)}(x,\xi,\Delta^2)] \\ &\quad + \frac{4M^2}{m_\pi^2-t} H_\pi^{(V)}\left(\frac{x}{\bar{p}_t\cdot n}, \frac{\xi}{\bar{p}_t\cdot n}, 0\right) \theta(\bar{p}_t\cdot n - |x|) \end{aligned} \tag{243}$$

$$H_2^{(-)} = -\frac{\tilde{H}^{(V)}(x,\xi_0,t)}{g_A} + H^{(V)}(x,\xi,\Delta^2) + E^{(V)}(x,\xi,\Delta^2) \tag{244}$$

$$H_3^{(-)} = -\left(\frac{M^2}{W^2-M^2} - \frac{M^2}{u-M^2}\right) E^{(V)}(x,\xi,\Delta^2) \tag{245}$$

$$H_4^{(-)} = -\left(\frac{M^2}{W^2-M^2} - \frac{M^2}{u-M^2}\right) [H^{(V)}(x,\xi,\Delta^2) + E^{(V)}(x,\xi,\Delta^2)]. \tag{246}$$

B.2 Isoscalar vector and gluon πN distributions

$$\begin{aligned} H_1^{(0,G)} &= E^{(S,G)}(x,\xi,\Delta^2) + \frac{2M^2\bar{\alpha}}{u-M^2}[H^{(S,G)}(x,\xi,\Delta^2) + E^{(S,G)}(x,\xi,\Delta^2)] \\ &\quad + \frac{2M^2}{m_\pi^2-t} H_\pi^{(S,G)}\left(\frac{x}{\bar{p}_t\cdot n}, \frac{\xi}{\bar{p}_t\cdot n}, 0\right) \theta(\bar{p}_t\cdot n - x) \end{aligned} \tag{247}$$

$$H_2^{(0,G)} = 0 \tag{248}$$

$$H_3^{(0,G)} = -\left(\frac{M^2}{W^2-M^2} + \frac{M^2}{u-M^2}\right) E^{(S,G)}(x,\xi,\Delta^2) \tag{249}$$

$$H_4^{(0,G)} = -\left(\frac{M^2}{W^2-M^2} + \frac{M^2}{u-M^2}\right) [H^{(0,G)}(x,\xi,\Delta^2) + E^{(0,G)}(x,\xi,\Delta^2)] \tag{250}$$

B.3 Axial vector πN distributions

$$\tilde{H}_1^{(0,+)} = \xi \tilde{E}^{(S,V)}(x,\xi,\Delta^2) - \frac{2M^2\bar{\alpha}}{u-M^2}\tilde{H}^{(S,V)}(x,\xi,\Delta^2) \tag{251}$$

$$\tilde{H}_2^{(0,+)} = 0 \tag{252}$$

$$\tilde{H}_3^{(0,+)} = -\left(\frac{M^2}{W^2-M^2} - \frac{M^2}{u-M^2}\right)\xi \tilde{E}^{(S,V)}(x,\xi,\Delta^2) \tag{253}$$

$$\tilde{H}_4^{(0,+)} = -\left(\frac{M^2}{W^2-M^2} - \frac{M^2}{u-M^2}\right)\tilde{H}^{(S,V)}(x,\xi,\Delta^2), \tag{254}$$

$$\tilde{H}_1^{(-)} = \frac{E^{(V)}(x,\xi_0,t)}{g_A} + \frac{2M^2\bar{\alpha}}{u-M^2}\tilde{H}^{(V)}(x,\xi,\Delta^2) \tag{255}$$

$$\tilde{H}_2^{(-)} = -\frac{E^{(V)}(x,\xi_0,t) + H^{(V)}(x,\xi_0,t)}{g_A} + \tilde{H}^{(V)}(x,\xi,\Delta^2) \tag{256}$$

$$\begin{aligned}\tilde{H}_3^{(-)} &= -\left(\frac{1}{2} + \frac{M^2}{W^2-M^2} + \frac{M^2}{u-M^2}\right)\xi \tilde{E}^{(V)}(x,\xi,\Delta^2) \\ &\quad + \frac{2M^2}{g_A(m_\pi^2-\Delta^2)}\,\phi_\pi\left(\frac{x}{\xi}\right)\,\theta(\xi-|x|)\end{aligned} \tag{257}$$

$$\tilde{H}_4^{(-)} = -\left(\frac{M^2}{W^2-M^2} + \frac{M^2}{u-M^2}\right)\tilde{H}(x,\xi,\Delta^2). \tag{258}$$

C Soft-pion theorems for the form factors of pion emission induced by the local vector current

In equation (38) of section 3.3 in the main text, we have parametrized the matrix element for pion emission induced by the local vector current in terms of form factors A_i. As explained in this section, the soft-pion theorems for these form factors can be obtained from the first moments of the πN distributions functions $H_i^{(0,\pm)}$ that have been given in appendix B. In this way, we obtain for the isoscalar and isovector even form factors:

$$\begin{aligned}&\sum_{i=1}^{8} U'\, A_i^{(0,+)}\Gamma_i^\mu U \\ &= \bar{U}'\left\{F_1^{(S,V)}(\Delta^2)\,\bar{p}^\mu + \frac{2M^2[F_1^{(S,V)}(\Delta^2)+F_2^{(S,V)}(\Delta^2)]}{u-M^2}k^\mu \right.\\ &\quad -\left(\frac{M}{W^2-M^2}+\frac{M}{u-M^2}\right)F_2^{(S,V)}(\Delta^2)\,\hat{k}\bar{p}^\mu \\ &\quad \left. -\left(\frac{M^2}{W^2-M^2}+\frac{M^2}{u-M^2}\right)[F_1^{(S,V)}(\Delta^2)+F_2^{(S,V)}(\Delta^2)]\hat{k}\gamma^\mu\right\}\gamma_5 U + \mathcal{O}(\varepsilon).\end{aligned} \tag{259}$$

To the given accuracy, current conservation is fulfilled:

$$\Delta^\mu \sum_i \bar{U}' A_i^{(0,+)} \Gamma_i^\mu U = \mathcal{O}(\varepsilon). \tag{260}$$

For the isovector odd form factors, we obtain

$$\begin{aligned}
&\sum_{i=1}^{8} U'\, A_i^{(-)} \Gamma_i^\mu U \\
&= \bar{U}' \Bigg\{ -\frac{G_P^{(V)}(t)}{2g_A} \Delta^\mu + \left[-\frac{2M^2 [F_1^{(V)}(\Delta^2) + F_2^{(V)}(\Delta^2)}{u - M^2} + \frac{4M^2}{m_\pi^2 - t} \right] k^\mu \\
&\quad + [-F_A^{(V)}(t) + F_1^{(V)}(\Delta^2) + F_2^{(V)}(\Delta^2)] M \gamma_\mu - \left(\frac{M}{W^2 - M^2} - \frac{M}{u - M^2} \right) F_2^{(V)}(\Delta^2)\, \hat{k} \bar{p}^\mu \\
&\quad - \left(\frac{M}{W^2 - M^2} - \frac{M}{u - M^2} \right) F_2^{(V)}(\Delta^2)\, \hat{k} \bar{p}^\mu \\
&\quad - \left(\frac{M^2}{W^2 - M^2} - \frac{M^2}{u - M^2} \right) [F_1^{(V)}(\Delta^2) + F_2^{(V)}(\Delta^2)] \hat{k} \gamma^\mu \Bigg\} \gamma_5 U + \mathcal{O}(\varepsilon).
\end{aligned} \tag{261}$$

Current conservation at small momentum transfer is immediately fulfilled, but for the moderate momentum transfer, the pseudoscalar form factor G_P cannot simply be approximated by its pion pole form, and so we arrive at

$$\Delta_\mu \sum_i \bar{U}' A_i^{(-)} \Gamma_i^\mu U = -\frac{M}{g_A} \left[G_A(t)\, 2M + G_P(t) \frac{t}{2M} \right] \bar{U}' \gamma_5 U + \mathcal{O}(\varepsilon) \qquad (-t \gg \varepsilon^2). \tag{262}$$

But since the remaining combination of axial and pseudoscalar form factor is proportional to m_π^2,

$$ß\bar{U}' \left[G_A(t)\, 2M + G_P(t) \frac{t}{2M} \right] \gamma_5 \frac{\tau^a}{2} U = \langle N(p')|\partial \cdot A^a|N(p)\rangle = f_\pi m_\pi^2 \langle N(p')|\Phi^a|N(p)\rangle \tag{263}$$

and pion-pole enhancement at such large t is excluded, it is reasonable to assume $G_A(t)\, 2M + G_P(t)\, t/(2M) = \mathcal{O}(\varepsilon)$. Hence for any momentum transfer in the considered region $-t \leq M^2$ we arrive at

$$\Delta^\mu \sum_i \bar{U}' A_i^{(-)} \Gamma_i^\mu U = \mathcal{O}(\varepsilon), \tag{264}$$

so that finally, current conservation is fulfilled.

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